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An explicit bound for the number of partitions into roots

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A B S T R A C T

We use the saddle point method to prove an explicit upper bound for the number of representations of a positive integer n into the form $\lfloor\sqrt{a_1}\rfloor + \lfloor\sqrt{a_2}\rfloor + \ldots + \lfloor\sqrt{a_k}\rfloor$, where k and $a_1, a_2, \ldots, a_k$ are positive integers. We also give an asymptotic formula for this number as $n \to \infty$.

1. Introduction

The number of representations of a positive integer n of the form $\lfloor\sqrt{a_1}\rfloor + \lfloor\sqrt{a_2}\rfloor + \cdots + \lfloor\sqrt{a_k}\rfloor$ with positive integers k and $a_1 \le a_2 \le \cdots \le a_k$ was introduced by Balasubramanian and Luca in [1]. In their paper, they investigated the size of the set

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$$\mathcal{F}(x) = \{\Delta(n) \le x\},$$

where $\Delta(n)$ is the number of unordered factorizations of the positive integer n into factors > 1. They proved that for all $x \ge 1$,

$$|\mathcal{F}(x)| \le \exp\left(9(\log x)^{2/3}\right). \tag{1}$$

Their proof of inequality (1) however depended on the following bound, which was also proved in the same paper:

$$q(n) \le \exp\left(5n^{2/3}\right), \tag{2}$$

where $q(n)$ denotes the number of representations of n of the form

$$n = \lfloor\sqrt{a_1}\rfloor + \lfloor\sqrt{a_2}\rfloor + \ldots + \lfloor\sqrt{a_k}\rfloor,$$

where k and $a_1 \le a_2 \le \cdots \le a_k$ are positive integers.

Chen and Li [2] found a gap in the proof of (2). They provided an alternative proof that gives a weaker upper bound as well as a lower bound for $q(n)$. Their result is the following:

$$\exp\left(c_1 n^{2/3}\right) \le q(n) \le \exp\left(c_2 n^{2/3}\right),$$

where $c_1 \approx 5.385 \times 10^{-24}$ and $c_2 \approx 22.962$. In this paper, we use the saddle point method to obtain the following explicit upper bound on $q(n)$:

Theorem 1. *For $n \ge 50$ we have*

$$\log q(n) \le \frac{6\zeta(3)^{1/3}}{4^{2/3}} n^{2/3} \left(1 + \frac{3.2}{\sqrt[3]{n+2}}\right), \tag{3}$$

with

$$\frac{6\zeta(3)^{1/3}}{4^{2/3}} \approx 2.532.$$

The condition $n \ge 50$ is only a technicality that we assume to obtain a relatively small constant in the second term of the bound in (3). For $n \le 50$, Fig. 1 shows that an even better upper bound holds (in Fig. 1 the curve represents the graph of the function $x \mapsto \frac{6\zeta(3)^{1/3}}{4^{2/3}} x^{2/3}$ and the dotted line represents the graph of the sequence of general term $\log q(n)$).

Note that if $n \ge 50$ then

$$\frac{6\zeta(3)^{1/3}}{4^{2/3}} \left(1 + \frac{3.2}{\sqrt[3]{n+2}}\right) \ge \frac{6\zeta(3)^{1/3}}{4^{2/3}} \left(1 + \frac{3.2}{\sqrt[3]{52}}\right) \approx 4.7,$$