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Almost prime triples and Chen's theorem

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We show that there are infinitely many primes p such that not only does $p+2$ have at most two prime factors, but $p+6$ also has a bounded number of prime divisors. This refines the well known result of Chen [2].

1. Introduction

The twin prime conjecture states that there are infinitely many primes p such that $p+2$ is also prime. Although the conjecture has resisted our efforts, there has been spectacular partial progress. One well known result is Chen's theorem [2] that there are infinitely many primes such that $p+2$ has at most two prime factors. In a different

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direction, building on the work of Goldston, Pintz, and Yıldırım [5], it has recently been shown by Zhang [11] that there are bounded gaps between consecutive primes infinitely often. The numerical result has been improved in the works of the Polymath8 project [9] and Maynard [8], and the bounded gaps result has also been extended to prime tuples by Maynard [8] and Tao (unpublished).

The twin prime conjecture is a special case of the Hardy–Littlewood conjecture, which postulates asymptotics for prime tuples in general. An example is that one expects that the number of primes $p \leq x$ such that $p+2$ and $p+6$ are simultaneously prime should be asymptotic to

$$C\frac{x}{\log^3 x}$$

for a certain positive constant C (given by (32)). In this direction, it has been proven that there are infinitely many natural numbers n such that $n(n+2)(n+6)$ is almost prime — that is, $n(n+2)(n+6)$ has at most r prime factors, for some finite r. More specifically, Porter [10] proved this statement for $r = 8$ and this was improved by Maynard [7] to $r = 7$.

We are interested in proving an analogue of Chen's theorem for prime tuples. More precisely, we show that there are infinitely many primes p such that $p+2$ has at most two prime factors, and $p+6$ has at most r prime factors for some finite r.

Theorem 1. *Let $\pi_{1,2,r}(x)$ denote the number of primes $p \leq x$ such that $p+2$ has at most two prime factors and $p+6$ has at most r prime factors. Then*

$$\pi_{1,2,r}(x) \gg \frac{x}{\log^3 x} \tag{1}$$

for $r = 76$.

Our basic philosophy, which the proof will illustrate, is the following. Suppose one has polynomials $f_1(x), \ldots, f_{k+1}(x)$ and positive integers $r_1, \ldots, r_k$. Then, if the weighted sieve can prove that

$$f_1(n) = P_{r_1}, \ldots, f_k(n) = P_{r_k}$$

for infinitely many integers n, then one should be able to modify the argument to show the existence of a positive integer r_{k+1} such that

$$f_1(n) = P_{r_1}, \ldots, f_{k+1}(n) = P_{r_{k+1}}$$

for infinitely many integers n.

Our approach uses the weighted sieve which appeared in Chen's original work, as well as the vector sieve of Brüdern and Fouvry [1]. We will also use a Selberg upper bound