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Quantitative versions of the joint distributions of Hecke eigenvalues

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A B S T R A C T

In 2009, Omar and Mazhouda proved that as $k \to \infty$, $\{\lambda_f(p^2) : f \in H_k\}$ and $\{\lambda_f(p^3) : f \in H_k\}$ are equidistributed with respect to some measures respectively, where H_k is the set of all the normalized primitive holomorphic cusp forms of weight k for $SL_2(\mathbb{Z})$. In this paper, we obtain a quantitative version of Omar and Mazhouda's result. Moreover, we find out that $\{\lambda_f(p^4) : f \in H_k\}$ and $\{\lambda_f(p^r) - \lambda_f(p^{r-2}) : f \in H_k$ and $r \geq 2\}$ follow some nice distribution laws respectively as $k \to \infty$ and get quantitative versions of these distributions. In the context of Maass cusp forms, we establish analogous results.

1. Introduction

Let H_k be the set of normalized primitive holomorphic cusp forms of even integral weight k for the modular group $\Gamma = SL_2(\mathbb{Z})$. Given any $f \in H_k$ and prime p, the

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distribution of the normalized Hecke eigenvalues $\lambda_f(p)$ is an interesting and difficult problem. The generalized Ramanujan conjecture for primitive holomorphic cusp forms implies that $|\lambda_f(p)| \leq 2$ for any $f \in H_k$ and prime p. This conjecture was proved by Deligne [4] in 1974.

Inspired by the Sato–Tate conjecture, Serre studied the asymptotic distribution of the Hecke eigenvalues $\lambda_f(p)$, as f is fixed and the primes p vary. In the 1960's, Serre conjectured that for any $f \in H_k$, as $x \to \infty$, $\lambda_f(p)$, $p \leq x$, are equidistributed in $[-2, 2]$ with respect to the Sato–Tate measure

$$d\mu_\infty(x) = \begin{cases} \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}\, dx & \text{if } x \in [-2, 2], \\ 0 & \text{otherwise,} \end{cases}$$

which is also called the Sato–Tate conjecture. This conjecture was proved by Barnet-Lamb, Geraghty, Harris and Taylor [1] in 2011.

For a fixed prime p, as $k \to \infty$, the values of $\lambda_f(p)$, $f \in H_k$, also follow some nice distribution laws. Conrey, Duke and Farmer [3] and Serre [11] figured out that they are equidistributed with respect to the p-adic measure

$$d\mu_p(x) = \begin{cases} \frac{p+1}{2\pi} \frac{\sqrt{4-x^2}}{(p^{1/2}+p^{-1/2})^2 - x^2}\, dx & \text{if } x \in [-2, 2], \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Moreover, Murty and Sinha [8] proved that the rate of convergence to the above distribution is $O\left(\frac{\log p}{\log k}\right)$. Lau and Wang [7] later generalized Murty and Sinha's result to a joint distribution.

On the other hand, it is well-known that for any $m, n \geq 1$

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

and so

$$\lambda_f(p^n) = X_n\left(\frac{\lambda_f(p)}{2}\right),$$

where X_n is the nth Chebychev polynomial of the second kind. One may naturally consider the distribution of $\{\lambda_f(p^n) : f \in H_k\}$ for some fixed prime p and as k goes to infinity. In this direction, Omar and Mazhouda [9] proved that $\{\lambda_f(p^2) : f \in H_k\}$ and $\{\lambda_f(p^3) : f \in H_k\}$ are equidistributed with respect to the measures

$$d\mu_{p,2}(x) = \begin{cases} m_{p,2}(x)\, dx & \text{if } x \in [-1, 3], \\ 0 & \text{otherwise,} \end{cases}$$

and