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Remarks on polygamma and incomplete gamma type functions

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A B S T R A C T

We give a meaning to the expression $\psi^{(n)}(-m)$ in neutrix setting. Further the incomplete gamma type function $\gamma_*(\alpha, x_-)$ is introduced for negative values of α.

0. Introduction

The polygamma function is defined by

$$\psi^{(n)}(x) = \frac{d^n}{dx^n}\psi(x) = \frac{d^{n+1}}{dx^{n+1}} \ln \Gamma(x) \qquad (x > 0), \tag{1}$$

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and appears in a number of theoretical and practical applications. Its integral representation is given by

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}$$

which holds for $x > 0$, and this can be also written as

$$\psi^{(n)}(x) = -\int_0^1 \frac{t^{x-1} \ln^n t}{1-t}\, dt. \tag{2}$$

The recursion formula

$$\psi^{(n)}(x+1) = \psi^{(n)}(x) + \frac{(-1)^n n!}{x^{n+1}}$$

can be used to define the polygamma function for negative non-integer values of x, see [8]. If $-r < x < -r+1$ $(r \in \mathbb{N})$, then we have

$$\psi^{(n)}(x) = -\int_0^1 \frac{t^{x+r-1}}{1-t} \ln^n t\, dt - \sum_{k=0}^{r-1} \frac{(-1)^n n!}{(x+k)^{n+1}}.$$

And by regularization

$$\begin{aligned}\psi^{(n)}(x) = &-\int_0^{1/2} t^{x-1} \ln^n t \Big[(1-t)^{-1} - \sum_{i=0}^{r-1} t^i\Big]\, dt + \\ &+ \int_{1/2}^1 t^{x-1} \ln^n t (1-t)^{-1}\, dt + \\ &- \sum_{i=0}^{r-1} \sum_{j=0}^{n} \frac{(-1)^n n! \ln^{n-j} 2}{(n-j)!(x+i)^{j+1} 2^{x+i}}\end{aligned} \tag{3}$$

for $x > -r$ and $x \neq 0, -1, -2, \dots, -r+1$, where the function $t^{x-1} \ln^n t(1-t)^{-1}$ converges to zero as $x \to 1$.

Using the concepts of neutrix and neutrix limit due to van der Corput, Fisher gave general principle for the discarding of unwanted infinite quantities from asymptotic expansions [3]. The following definitions were given by van der Corput [2].

A neutrix N is defined as a commutative additive group of functions $\nu(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some ν in N, $\nu(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in N are called negligible functions.