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# Truncated series from the quintuple product identity

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ABSTRACT

We examine two truncated series derived from the quintuple product identity and prove that one has nonnegative coefficients and the other has nonpositive coefficients. In addition, we show that truncated series arising from two well-known consequences of the quintuple product identity also have such nonnegativity.



## 1. Introduction

Let $p(n)$ denote the number of partitions of $n$. In a recent paper [8], M. Merca proved the inequality

$$p(n) - p(n-1) - p(n-2) + p(n-5) \leq 0, \tag{1.1}$$

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in order to provide a fast algorithm generating the partitions of $n$. Subsequently, G.E. Andrews and Merca questioned the existence of an infinite family of inequalities to which (1.1) belongs and provided their answer in [3]. In particular, they proved that for integers $n \geq 1$ and $k \geq 1$,

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\left(p\left(n-\frac{j(3j+1)}{2}\right)-p\left(n-\frac{j(3j+5)}{2}-1\right)\right)\geq 0. \tag{1.2}$$

One can easily verify that (1.2) reduces to (1.1) when $k = 2$.

Motivated by the results in [3], V.J.W. Guo and J. Zeng proved truncated versions of two identities by Gauss in [6] and made a conjecture which is similar to (1.2). In particular, let the generating function of $t(n)$ be as follows

$$\sum_{n=0}^{\infty} t(n)q^n := \frac{1}{(q;q)_\infty^3}.$$

Guo and Zeng [6] conjectured that for $n \geq 1$ and $k \geq 1$, we have

$$(-1)^k\sum_{j=0}^{k}(-1)^j(2j+1)t\left(n-\frac{j(j+1)}{2}\right)\geq 0.$$

This was later proved by the third author in [7].

Furthermore, while working on (1.2), Andrews and Merca conjectured that for $1 \leq S < R/2$ and $k \geq 1$, the truncated series

$$\frac{1}{(q^S,q^{R-S},q^R;q^R)_\infty}\sum_{n=0}^{k-1}(-1)^n q^{n(n+1)R/2-nS}(1-q^{(2n+1)S}) \tag{1.3}$$

has nonnegative coefficients if $k$ is odd, and nonpositive coefficients if $k$ is even [3]. It is noted that series (1.3) arises from the following specialization of the Jacobi triple product identity. For $1 \leq S < R/2$, we have

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)R/2-nS}(1-q^{(2n+1)S}) = (q^S,q^{R-S},q^R;q^R)_\infty.$$

Their conjecture was proved independently by A.J. Yee using combinatorial methods [9], and by the third author via partial theta functions [7].

The quintuple product identity can be considered as the next product-series identity after the Jacobi triple product identity. It is most commonly stated in the form (see [4, pp. 118–119])

$$\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}(x^{3n}-x^{-3n-1}) = (x^{-1},xq,q;q)_\infty(x^{-2}q,x^2q;q^2)_\infty. \tag{1.4}$$