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Newforms for odd orthogonal groups

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ABSTRACT

I introduce a new notion of newforms for split special odd orthogonal groups which generalizes that of classical newforms by defining a family of compact subgroups of their adelic groups.

1. Introduction

The theory of newforms has long been a fascinating subject in number theory. It originated from the Shimura–Taniyama Conjecture (or the Modularity Theorem), which associates elliptic curves over $\mathbb{Q}$ with classical holomorphic modular forms. Weil refined the conjecture and gave a precise description on such a modular form whose level equals the conductor of the elliptic curve. It comes naturally to ask if such a refinement exists for general motives; namely, if in the conjectural Langlands program we can also specify

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a particular automorphic form in the automorphic representation associated with the motive.

After his 60th birthday conference in 2010, Gross [8] wrote a letter to Serre on a conjectural construction of such an automorphic form on special odd orthogonal group to a pure symplectic motive. He proposed the existence of a distinguished line in each generic local representation. It was verified that this notion generalized the newforms for GL_2 introduced by Atkin–Lehner and Li in early 70s. He proved that such a line exists at the real place assuming the representation is a generic limit discrete series. He discussed a suggestion on the finite places by Brumer which is later proved by the author as her PhD thesis under his supervision.

In this article we review the classical theory for $\mathrm{PGL}_2\,(\simeq \mathrm{SO}_3)$ and present a conjectural framework for newforms of odd special orthogonal group with which symplectic motives are associated in the Langlands program, and describe corresponding local result in a real place and finite places.

2. Atkin–Lehner–Li theory

Consider the space of classical holomorphic cusp forms $S_k(\Gamma_0(N))$ of level N weight k. The Hecke algebra consists of averaging operators on $S_k(\Gamma_0(N))$ generated by the characteristic functions T_n of the double coset

$$\Gamma_0(N)\left(\begin{smallmatrix}1&0\\0&n\end{smallmatrix}\right)\Gamma_0(N)$$

for all $n\in\mathbb{N}$. Atkin and Lehner [1] showed that there exists a basis of $S_k(\Gamma_0(N))$ consisting of simultaneous eigenforms of T_n for all $(n,N)=1$. Among these the newforms, opposed from oldforms, can be shown to be eigenforms of T_n for all n, and each corresponds to a unique collection of eigenvalues $\{a_n\}_{n\in\mathbb{N}}$. The eigenvalue a_n of T_n agrees with the n-th Fourier coefficient for an eigenform. The set of Fourier coefficients uniquely determines the holomorphic newform.

These conclude to a characterization of holomorphic newforms that among all nonzero eigenforms with the same collection of eigenvalues a_p for all prime number p, $(p,N)=1$, the holomorphic newforms have the smallest level and the constant Fourier coefficients are nonzero and normalized to 1. The characterization automatically implies the sets of Fourier coefficients uniquely determine the holomorphic newforms and these newforms are eigenforms for all Hecke operators T_n, $n\in\mathbb{N}$.

Whenever a holomorphic newform f of weight k level N is given, the Fourier coefficients a_n's of f satisfy good recurrence relations gotten from the action of the Hecke operators. The theory of Atkin–Lehner [1] and Li [12] gives that the L-function

$$L(s,f)=\sum_{n=1}^{\infty}a_n n^{-s}$$