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Generalized Legendre curves and quaternionic multiplication

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Dedicated to Professor Wen-Ching Winnie Li, an incredible woman, an amazing leader, who inspired so many of us

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A B S T R A C T

This paper is devoted to abelian varieties arising from generalized Legendre curves. In particular, we consider their corresponding Galois representations, periods, and endomorphism algebras. For certain one parameter families of 2-dimensional abelian varieties of this kind, we determine when the endomorphism algebra of each fiber defined over the algebraic closure of $\mathbb{Q}$ contains a quaternion algebra.

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Counting points over finite fields
Shimura curves

1. Introduction

Algebraically, for integers $2 \leq e_1, e_2, e_3 \leq \infty$, the *triangle group* (e_1, e_2, e_3) is defined by the presentation

$$\langle x, y \mid x^{e_1} = y^{e_2} = (xy)^{e_3} = id \rangle.$$

A triangle group is called *arithmetic* if it has a unique embedding to $SL_2(\mathbb{R})$ with image either commensurable with $PSL_2(\mathbb{Z})$, which is isomorphic to $(2, 3, \infty)$, or derived from an order of certain indefinite quaternion algebra over a totally real field, see [31, Section 3, Defn. 1]. Arithmetic triangle groups have been classified by Takeuchi [31,32]. Given an arithmetic triangle group Γ, its action on the upper half plane, $\mathfrak{H}$, via linear transformations yields a quotient space. This space is a modular curve when at least one of e_i is ∞; otherwise, it is a Shimura curve. While modular curves parametrize certain isomorphism classes of elliptic curves, Shimura curves parametrize isomorphism classes of certain 2-dimensional abelian varieties with quaternionic multiplication [28], [29]. For this reason, our main result focuses on 2-dimensional abelian varieties. Recently, Yang has computed explicitly automorphic forms on Shimura curves in terms of hypergeometric series [39]. Based on that, Yang and the fifth author obtained explicit algebraic transformations of hypergeometric functions [33].

Each arithmetic triangle group can be realized as a monodromy group of some ordinary differential equation (ODE) satisfied by an integral of the form

$$\int_0^1 \frac{dx}{\sqrt[N]{x^i(1-x)^j(1-\lambda x)^k}}, \tag{1}$$

with $N, i, j, k \in \mathbb{Z}$. Wolfart realized these integrals as periods of the generalized Legendre curves

$$C_\lambda^{[N;i,j,k]} : y^N = x^i(1-x)^j(1-\lambda x)^k,$$

where λ is a constant and N, i, j, k are suitable natural numbers [35]. In this paper, we assume that $1 \leq i, j, k < N$. We additionally assume $N \nmid i + j + k$, so that the generic abelian subvarieties that we extract from the smooth models of $C_\lambda^{[N;i,j,k]}$ do not have complex multiplication (CM). For a g-dimensional abelian variety A over a totally real field K, this means the endomorphism of A is a degree $2g$ CM field over K. The CM cases have been previously considered in papers like [27] and will not be our focus here.

Let Γ be an arithmetic triangle group with each e_i finite. Then Γ corresponds to an explicit quaternion algebra H_Γ over a totally real field K_Γ. Takeuchi gives descriptions of