



ELSEVIER

Contents lists available at ScienceDirect

Journal of Number Theory

[www.elsevier.com/locate/jnt](http://www.elsevier.com/locate/jnt)



# Ternary quadratic forms over number fields with small class number



Markus Kirschmer<sup>\*,1</sup>, David Lorch

*Lehrstuhl B für Mathematik, RWTH Aachen University, 52062 Aachen, Germany*

## ARTICLE INFO

### Article history:

Received 22 September 2014

Received in revised form 25

November 2014

Accepted 26 November 2014

Available online 5 December 2014

Communicated by Jerome Hoffman,

Robert Perlis, Ling Long, Karl

Mahlburg, Jorge Morales, Holly

Swisher

### MSC:

primary 11E41

secondary 16H10

### Keywords:

Quadratic form

Genus

Class number

Quaternion order

## ABSTRACT

We enumerate all positive definite ternary quadratic forms over number fields with class number at most 2. This is done by constructing all definite quaternion orders of type number at most 2 over number fields. Finally, we list all definite quaternion orders of ideal class number 1 or 2.

© 2014 Elsevier Inc. All rights reserved.

\* Corresponding author.

*E-mail addresses:* [markus.kirschmer@math.rwth-aachen.de](mailto:markus.kirschmer@math.rwth-aachen.de) (M. Kirschmer), [david.lorch@math.rwth-aachen.de](mailto:david.lorch@math.rwth-aachen.de) (D. Lorch).

<sup>1</sup> The author has been supported by DFG grant KI 1594/1-1.

## 1. Introduction

The question of finding all positive definite quadratic forms with small class number dates back to Gauß. The binary case (where at present, a complete unconditional classification is out of reach) is related to relative ideal class numbers of CM-fields. Over the rationals, one-class genera cannot exist in dimension  $\geq 11$  (cf. [36]). The rational one-class genera have been studied extensively by Watson, see [35,37] and the references therein. He classified all such genera in three and more than five variables, and produced partial results in four and five variables. The authors have recently reinvestigated Watson's classification and filled in the details for the missing dimensions four and five (see [14]). An overview of the enumeration of genera with small class number is given in [30].

In the case of an arbitrary totally real number field, Pfeuffer [26] showed that one-class genera of positive definite quadratic forms cannot exist in more than 32 variables. The maximal integral forms with class number one have been enumerated recently by the first author in [13]. Although one expects very few examples of one-class genera of positive definite quadratic forms over totally real number fields in dimension  $\geq 5$ , no complete classification is known.

Pfeuffer's results give an upper bound on the local factors occurring in Siegel's mass formula, thus effectively bounding the discriminants of possible base fields for one-class genera. For non-maximal forms in dimension 3, these bounds are not quite sharp enough to yield the possible base fields using the currently available tables of totally real number fields.

The present article addresses this shortcoming by employing the correspondence of Brzezinski–Peters–Eichler–Brandt (see Section 3), which relates these genera to quaternion orders with type number at most 2 (we refer to Section 2 for definitions). Thus, we will enumerate all genera of positive definite ternary quadratic forms with class number at most 2 over any totally real number field. These Gorenstein orders can be enumerated directly using *Magma* [1]. This classification extends the work of Brzezinski [7] who computed the quaternion orders of type number one over the rationals.

It turns out that there are 4194 one-class genera of positive definite ternary quadratic forms over 30 different base fields. The largest base field has degree 5. Similarly, there are 18,538 two-class genera over 75 different base fields, the largest one of which has degree 6.

This article is organized as follows. In Section 2, we recall properties of quaternion algebras and orders. In Section 3, we discuss a correspondence between quaternion orders and lattices in definite quadratic spaces. The list of all definite hereditary quaternion orders of type number at most 2 will be computed in Section 4. In Section 5 we extend this list to all definite quaternion orders of type number at most 2. Finally, in the last section we enumerate all definite quaternion orders having ideal class number at most 2.

A complete list of these orders and genera can be obtained electronically from [15].

Download English Version:

<https://daneshyari.com/en/article/4593227>

Download Persian Version:

<https://daneshyari.com/article/4593227>

[Daneshyari.com](https://daneshyari.com)