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On the Eisenstein ideal over function fields

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ABSTRACT

We study the Eisenstein ideal of Drinfeld modular curves of small levels, and the relation of the Eisenstein ideal to the cuspidal divisor group and the component groups of Jacobians of Drinfeld modular curves. We prove that the characteristic of the function field is an Eisenstein prime number when the level is an arbitrary non-square-free ideal of $\mathbb{F}_q[T]$ not equal to a square of a prime.

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1. Introduction

The Eisenstein ideal for modular curves over $\mathbb{Q}$ was introduced by Mazur in his seminal paper [19], and since then the Eisenstein ideal has become an indispensable tool in various problems related to modular curves, modular Jacobians, modular Galois representations, etc. The problem to develop the theory of Eisenstein ideals for Drinfeld modular curves was suggested by Mazur, already in the introduction of [19]. The first attempt to develop this theory was made by Tamagawa [32], but more comprehensive results were obtained by Pál [21]. Both [32] and [21] assume that the level is prime. In [27], in connection with the problem of Jacquet–Langlands isogenies over function fields, we examined the Eisenstein ideal on Drinfeld modular curves whose level is a product of two distinct primes. We discovered that some of the properties of the Eisenstein ideal in that case are quite different from its prime level counterpart. In this paper we continue our study of the Eisenstein ideal for non-prime levels, and its relation to the cuspidal divisor group and the component groups of Jacobians of Drinfeld modular curves. Our goal here is to compute everything explicitly when the level is small, and from this make some predictions about the behaviour of the Eisenstein ideal in general.

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a power of a prime number p. Let $A = \mathbb{F}_q[T]$ be the ring of polynomials in indeterminate T with coefficients in $\mathbb{F}_q$, and $F = \mathbb{F}_q(T)$ be the rational function field. The degree map $\deg : F \to \mathbb{Z} \cup \{-\infty\}$, which associates to a non-zero polynomial its degree in T and $\deg(0) = -\infty$, defines a norm on F by $|a| := q^{\deg(a)}$. The corresponding place of F is usually called the *place at infinity*, and is denoted by ∞; it plays a role similar to the archimedean place of $\mathbb{Q}$. We also define a norm and degree on the ideals of A by $|\mathfrak{n}| := \#(A/\mathfrak{n})$ and $\deg(\mathfrak{n}) := \log_q |\mathfrak{n}|$. Let F_∞ denote the completion of F at ∞, and $\mathbb{C}_\infty$ denote the completion of an algebraic closure of F_∞. Let $\Omega := \mathbb{C}_\infty - F_\infty$ be the *Drinfeld half-plane*.

Let $\mathfrak{n} \lhd A$ be a non-zero ideal. The level-$\mathfrak{n}$ *Hecke congruence subgroup* of $\mathrm{GL}_2(A)$ is

$$\Gamma_0(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \;\middle|\; c \equiv 0 \bmod \mathfrak{n} \right\}.$$

Let $\mathbb{T}(\mathfrak{n})$ be the $\mathbb{Z}$-algebra generated by the Hecke operators $T_\mathfrak{m}$, $\mathfrak{m} \lhd A$, acting on the group $\mathcal{H}_0(\mathfrak{n}, \mathbb{Z})$ of $\mathbb{Z}$-valued $\Gamma_0(\mathfrak{n})$-invariant cuspidal harmonic cochains on the Bruhat–Tits tree $\mathscr{T}$ of $\mathrm{PGL}_2(F_\infty)$; see Section 2 for the definitions. The *Eisenstein ideal* $\mathfrak{E}(\mathfrak{n})$ of $\mathbb{T}(\mathfrak{n})$ is the ideal generated by the elements

$$\left\{ T_\mathfrak{p} - |\mathfrak{p}| - 1 \;\middle|\; \mathfrak{p} \text{ is prime, } \mathfrak{p} \nmid \mathfrak{n} \right\}.$$

(For some alternative ways of defining this ideal see Section 5.6.) The quotient ring $\mathbb{T}(\mathfrak{n})/\mathfrak{E}(\mathfrak{n})$ is finite (Lemma 2.9), and constitutes the main object of study of this paper. In some sense, $\mathbb{T}(\mathfrak{n})/\mathfrak{E}(\mathfrak{n})$ encodes congruences between cuspidal harmonic cochains and Eisenstein series.