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Euler sums and integrals of polylogarithm functions

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ABSTRACT

This paper develops an approach to evaluation of Euler sums and integrals of polylogarithm functions. The approach is based on simple Cauchy product formula computations. Using the approach, some relationships between Euler sums and integrals of polylogarithm functions are established. A kind of seven, eight and nine order sums of Euler sums are obtained. Furthermore, we give explicit formula for several classes of Euler sums and integrals of polylogarithm functions in terms of Riemann zeta values.

1. Introduction

Harmonic numbers, alternating harmonic numbers and their generalizations in the paper are classically defined by

$$H_n = \sum_{j=1}^{n} \frac{1}{j},\ \zeta_n(k) = \sum_{j=1}^{n} \frac{1}{j^k},\ L_n(k) = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j^k},\ 1 \le k \in Z. \tag{1.1}$$

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The subject of this paper is Euler sums and integrals of polylogarithm functions. Euler sums are the infinite sums whose general term is a product of harmonic numbers and alternating harmonic numbers of index n and a power of n^{-1}. So, more generally we can define the nonlinear Euler sums by the series

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{m_1}\zeta_n^{q_i}(k_i)\prod_{j=1}^{m_2}L_n^{l_j}(h_j)}{n^p},\quad \sum_{n=1}^{\infty}\frac{\prod_{i=1}^{m_1}\zeta_n^{q_i}(k_i)\prod_{j=1}^{m_2}L_n^{l_j}(h_j)(-1)^{n-1}}{n^p},\tag{1.2}$$

where $p>1, m_1, m_2, q_i, k_i, h_j, l_j$ are positive integers. If $\sum_{i=1}^{m_1}(k_iq_i)+\sum_{j=1}^{m_2}(h_jl_j)+p=C$ (C is a positive integer), then we call it C-order Euler sums.

Apart from the actual evaluation of the series, one of the main questions that one sets out to solve is whether or not a given series can be expressed in terms of a linear rational combination of known constants. When this is the case, we say that the series is reducible to these values.

It has been discovered in the course of the years that many Euler sums admit expressions involving finitely the zeta values, that is to say values of the Riemann zeta function,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \Re(s)>1$$

with positive integer arguments. The general multiple zeta function is defined as

$$\zeta(s_1,s_2,\cdots,s_m)=\sum_{n_1>n_2>\cdots>n_m>0}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_m^{s_m}},$$

where $s_1+\cdots+s_m$ is called the weight and m is the multiplicity.

For a pair (p,q) of positive integers with $q\geq 2$, the classical linear Euler sum is defined by

$$S_{p,q}=\sum_{n=1}^{\infty}\frac{1}{n^q}\sum_{k=1}^{n}\frac{1}{k^p}.\tag{1.3}$$

In 1742, Goldbach proposed to Euler the problem of expressing the $S_{p,q}$ in terms of values at positive integers of the Riemann zeta function $\zeta(s)$. Euler showed this problem in the case $p=1$ and gave a general formula for odd weight $p+q$ without any proof in 1775. The relationship between the values of the Riemann zeta function and Euler sums has been studied by many authors, for example see [1–7,9–13]. Philippe Flajolet and Bruno Salvy (see [9]) made use of contour integrals to obtain some explicit formula for several classes of Euler sums