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Some identities involving certain Hardy sum and Kloosterman sum ☆

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ABSTRACT

Text. By using the properties of Gauss sums and the mean value theorem of the Dirichlet L-function, a hybrid mean value problem involving certain Hardy sum and Kloosterman sum is studied. Two exact computational formulae are given, through which the cancellation phenomenon is revealed.

Video. For a video summary of this paper, please visit https://youtu.be/d-X81xErU7Q.

1. Introduction

For any positive integer k and an arbitrary integer h, the classical Dedekind sum $S(h, k)$ is defined by

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$$S(h,k)=\sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

The sum $S(h,k)$ plays an important role in the transformation theory of the Dedekind η function. The various properties of $S(h,k)$ were investigated by many authors. For example, L. Carlitz [2] obtained a reciprocity theorem of $S(h,k)$. J.B. Conrey et al. [3] studied the mean value distribution of $S(h,k)$, and proved an interesting asymptotic formula for $\sum_{h=1}^{k}{}' |S(h,k)|^{2m}$, where $\sum{}'$ denotes the summation over all $1 \le h \le k$ such that $(h,k)=1$.

G.H. Hardy [4] introduced another sum analogous to Dedekind sum as

$$S_5(h,k)=\sum_{j=1}^{k}(-1)^{j+[\frac{hj}{k}]}\left(\left(\frac{j}{k}\right)\right),$$

where h and k are integers with $k>0$, which was one of the well-known so called Hardy sums or Berndt's arithmetic sums. Actually G.H. Hardy used $S_5(h,k)$ to arouse the theory of $r_s(n)$, the number of representations of n as the sum of s squares. He [4] obtained formulae $r_s(n)$, for $5 \le s \le 8$ and asymptotic formulae for $s>8$. And R. Sitaramachandrarao [7] gave the relations between Hardy sums involving $S_5(h,k)$ and classical Dedekind sum $S(h,q)$ as the following:

Proposition 1.1. *Let $(h,k)=1$. If $h+k$ is even, then we have*

$$S_5(h,k)=-10S(h,k)+2S(2h,k)+4S(h,2k).$$

B.C. Berndt [1] and R. Sitaramachandrarao [7] also found the reciprocity theorem of $S_5(h,k)$ as:

Proposition 1.2. *Let h and k be coprime positive integers. If h and k are odd, then we have*

$$S_5(h,k)+S_5(k,h)=\frac{1}{2}-\frac{1}{2hk}.$$

Y. Simsek [5] established some relations among theta-functions, Hardy sums and Lambert series. Moreover, for each pair (h,k) of relatively prime integers with positive k, he [6] defined a new sum $Y(h,k)$ as