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Linear and algebraic independence of generalized Euler–Briggs constants

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ABSTRACT

Possible transcendental nature of Euler's constant γ has been the focus of study for sometime now. One possible approach is to consider γ not in isolation, but as an element of the infinite family of generalized Euler–Briggs constants. In a recent work [6], it is shown that the infinite list of generalized Euler–Briggs constants can have at most one algebraic number. In this paper, we study the dimension of spaces generated by these generalized Euler–Briggs constants over number fields. More precisely, we obtain non-trivial lower bounds (see Theorem 5 and Theorem 6) on the dimension of these spaces and consequently establish the infinite dimensionality of the space spanned. Further, we study linear and algebraic independence of these constants over the field of all algebraic numbers.

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1. Introduction

In 1731, Euler introduced the following constant

$$\gamma := \lim_{x\to\infty} (\sum_{n\le x} \frac{1}{n} \; - \; \log x)$$

and derived a number of identities involving γ, special values of the Riemann zeta function and other known constants. After Euler several other notable mathematicians including Gauss and Ramanujan have studied this constant in depth. For a beautiful account of the various aspects of research about this constant, we refer the reader to a recent article of Lagarias [12].

The appearance of γ in its various avatars makes it a fundamental object of study in number theory. Though we expect that γ is transcendental, it is not even known to be irrational. However there are some transcendence results involving γ. To the best of our knowledge, the first such result was due to Mahler [14]. He proved that for any non-zero algebraic number α, the number

$$\frac{\pi Y_0(\alpha)}{2J_0(\alpha)} - \log\frac{\alpha}{2} - \gamma$$

is transcendental, where J_0 and Y_0 are Bessel functions of the first and second kind of order zero. More precisely,

$$J_0(\alpha) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}(\frac{\alpha}{2})^{2n} \;, \quad H_n := \sum_{j=1}^{n}\frac{1}{j}$$

$$\text{and} \quad \frac{\pi}{2}Y_0(\alpha) := \left(\log(\frac{\alpha}{2}) + \gamma\right) J_0(\alpha) \; + \; \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{(n!)^2}(\frac{\alpha^2}{4})^n.$$

In the rather difficult subject of transcendence, sometimes it is more pragmatic to look at a family of numbers as opposed to a single number and derive something meaningful. There are two significant results which are worth mentioning at this point. The first one is by R. Murty and Saradha [17]. They proved the following theorem.

Theorem 1 *(Murty and Saradha). Let $q > 1$ be a natural number. At most one of the numbers*

$$\gamma, \;\gamma(a,q), 1 \le a \le q, \;(a,q) = 1$$

is algebraic. Here

$$\gamma(a,q) := \lim_{x\to\infty} (\sum_{\substack{n\le x \\ n\equiv a \bmod q}} \frac{1}{n} - \frac{1}{q}\log x).$$