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Modular forms of arbitrary even weight with no exceptional primes

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ABSTRACT

A result of Dieulefait–Wiese proves the existence of modular eigenforms of weight 2 for which the image of every associated residual Galois representation is as large as possible. We generalize this result to eigenforms of general even weight $k \geq 2$.

1. Introduction

The purpose of this note is to provide a modest generalization of a theorem of Dieulefait–Wiese. Before stating the result, we briefly recall some terminology and notation.

Let $f = \sum a_n q^n \in S_k(\Gamma_0(N))$ be a normalized cuspidal modular eigenform (henceforth simply called an "eigenform") of weight $k \geq 2$ and level $\Gamma_0(N)$ for some integer $N \geq 1$. Let $G_{\mathbf{Q}}$ denote the absolute Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. The Fourier coefficients $\{a_i\}$ generate a number field K_f. Let $\mathcal{O}_f$ be the ring of integers of K_f, let λ be a maximal ideal in $\mathcal{O}_f$ with residue characteristic ℓ, and write $\mathbf{F}_\lambda$ for the extension of $\mathbf{F}_\ell$ generated by $\{a_i \bmod \lambda\}$, the residues of the Hecke eigenvalues. By work of Deligne, there is a Galois representation

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$$\rho_{f,\lambda} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O}_{f,\lambda})$$

as well as an associated semisimple residual representation

$$\bar{\rho}_{f,\lambda} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_\lambda).$$

These representations are unramified outside the primes dividing $N\ell\infty$, and $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for almost all primes λ. Upon composing $\bar{\rho}_{f,\lambda}$ with the natural projection $\mathrm{GL}_2(\mathbf{F}_\lambda) \to \mathrm{PGL}_2(\mathbf{F}_\lambda)$, we obtain the projective representation

$$\bar{\rho}^{\mathrm{proj}}_{f,\lambda} : G_{\mathbf{Q}} \to \mathrm{PGL}_2(\mathbf{F}_\lambda).$$

By a result of Ribet [11, Theorem 3.1], if f does not have complex multiplication (CM), then the image of $\bar{\rho}^{\mathrm{proj}}_{f,\lambda}$ is "as large as possible" for all but finitely many primes λ. More precisely, for almost all λ, the image of $\bar{\rho}^{\mathrm{proj}}_{f,\lambda}$ is either $\mathrm{PGL}_2(\mathbf{F}_\lambda)$ or $\mathrm{PSL}_2(\mathbf{F}_\lambda)$ (see also [4, Corollary 3.2]). In Section 1.1 we briefly discuss the history of such results.

Definition 1. A maximal ideal λ of $\mathcal{O}_f$ is called *exceptional* if the image of $\bar{\rho}^{\mathrm{proj}}_{f,\lambda}$ is not $\mathrm{PGL}_2(\mathbf{F}_\lambda)$ or $\mathrm{PSL}_2(\mathbf{F}_\lambda)$. We may also say that $\bar{\rho}^{\mathrm{proj}}_{f,\lambda}$ is exceptional.

Remark 1. Recall that by Dickson's classification, if $\bar{\rho}_{f,\lambda}$ is both irreducible and exceptional, then the image must be either dihedral or isomorphic to A_4, S_4, or A_5.

Thus Ribet's theorem states that if f does not have CM, then it has only finitely many exceptional primes. The following theorem was proved by Dieulefait–Wiese.

Theorem 1. *(See [4, Theorem 6.2].) There exist eigenforms $(f_n)_{n\in\mathbf{N}}$ of weight 2 such that*

1. *for all n the eigenform f_n has no exceptional primes, and*
2. *for a fixed prime ℓ, the size of the image of $\bar{\rho}_{f_n,\lambda_n}$ for $\lambda_n \lhd \mathcal{O}_{f_n}$ is unbounded for running n.*

Remark 2. The eigenforms f_n in Theorem 1 have some additional technical properties. First, they do not have CM, which is a necessary condition. Second, they have no nontrivial inner twists; this is important for their application to the inverse Galois problem in [4]. While the modular forms which we construct in Theorem 2 also enjoy these properties, we will not mention them for the sake of brevity and ease of exposition.

In this paper, we modify the arguments of [4] to obtain a version of Theorem 1 for eigenforms of general even weight $k \geq 2$. The main result of this paper is the following.

Theorem 2. *Let $k \geq 2$ be an even integer. There exist eigenforms $(f_n)_{n\in\mathbf{N}}$ of weight k such that*