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Class numbers of quadratic Diophantine equations

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ABSTRACT

In this paper, the class number of a quadratic Diophantine equation is defined so that it can be viewed as a measure of the obstruction of the local-global principal for quadratic Diophantine equations, and we also give a method for computing the class number of quadratic Diophantine equations by using the mass formula.

1. Introduction

The arithmetic theory of quadratic forms is a classical topic and has been extensively studied by various methods. It is a natural question to extend such theory to more general inhomogeneous quadratic polynomials. Such generalization has already been considered by van der Blij in [15], Kneser in [5] and Watson in [16]. Recently, for various purposes, this topic was picked up by Shimura in [9], Sun in [13], Colliot-Thélène and Xu in [2], Wei and Xu in [17] and Chan and Oh in [1] and so on. In this paper, we give a method

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for computing the class number of inhomogeneous quadratic polynomials by using the mass formula.

Notation and terminology are standard. Let F be an algebraic number field, O_F be the ring of integers of F and Ω_F be the set of all non-trivial primes on F. For each $p \in \Omega_F$, F_p is the completion of F at p. Denote ∞_F as the set of all archimedean primes on F and $p < \infty_F$ for $p \in \Omega_F \backslash \infty_F$. For each $p < \infty_F$, we let O_{F_p} stand for the valuation ring of F_p and $O_{F_p}^{\times}$ for the group of units. Write $O_{F_p} = F_p$ for $p \in \infty_F$. Let $\mathbf{A}_F$ be the adèle group of F equipped with its restricted product topology.

An n-ary non-degenerate quadratic Diophantine equation in the variables $x_1, \cdots, x_n$ over F is an inhomogeneous quadratic polynomial of the form

$$f(\mathbf{x}) = \sum_{1 \le i,j \le n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + c \quad (a_{ij} = a_{ji})$$

with coefficients a_{ij}, b_i and c in F for all $1 \le i, j \le n$ and $det(a_{ij})_{1 \le i,j \le n} \ne 0$.

Two n-ary non-degenerate quadratic Diophantine equations $f(\mathbf{x})$ and $g(\mathbf{y})$ over F are called equivalent over O_F if there is an affine transformation over O_F

$$\mathbf{y} = A\mathbf{x} + \beta$$

with $A \in GL_n(O_F)$ and $\beta \in O_F^n$ such that $f(\mathbf{x}) = g(A\mathbf{x} + \beta)$ and denoted by $f(\mathbf{x}) \sim_{O_F} g(\mathbf{y})$.

One can generalize the concept of genus and class of quadratic forms to quadratic Diophantine equations.

Definition 1.1. The genus of a quadratic Diophantine equation $f(\mathbf{x})$ over F is defined as

$$gen(f(\mathbf{x})) = \{g(\mathbf{y}) : g(\mathbf{y}) \sim_{O_{F_p}} f(\mathbf{x}) \text{ for all } p < \infty_F\}.$$

The class of a quadratic Diophantine equation $f(\mathbf{x})$ over F is defined as

$$cls(f(\mathbf{x})) = \{g(\mathbf{y}) : g(\mathbf{y}) \sim_{O_F} f(\mathbf{x})\}.$$

The number of classes in $gen(f(\mathbf{x}))$ is called the class number of $f(\mathbf{x})$.

One can formulate the above concepts in more intrinsic language and more general setting.

Let V be a non-degenerate n-dimensional quadratic space over F with the symmetric bilinear map

$$B : V \times V \longrightarrow F \text{ with } q(x) = B(x, x)$$

for any $x \in V$ and the special orthogonal group