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Orders in cubic number fields

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ABSTRACT

For any cubic number field K and any conductor ideal $\mathfrak{f}$ of K we describe how to find all orders of K with conductor $\mathfrak{f}$. The result depends only on the factorization of the rational prime numbers in K, and it also yields an elementary proof for a formula for the zeta function of orders in a cubic number field.

1. Introduction and main results

Throughout this paper let K denote an algebraic number field of degree $[K:\mathbb{Q}] = n$ and $\mathcal{O}_K$ its ring of integers. For any non-zero ideal $\mathfrak{a} \triangleleft \mathcal{O}_K$ let $\mathcal{N}(\mathfrak{a}) \in \mathbb{N}$ denote its norm.

Recall that an *order* $\mathcal{O}$ *of* K is a subring $\mathcal{O} \subset \mathcal{O}_K$ whose quotient field equals K, and the largest ideal $\mathfrak{f} \triangleleft \mathcal{O}_K$ with $\mathfrak{f} \subset \mathcal{O}$ is called the *conductor of* $\mathcal{O}$. We call a non-zero ideal $\mathfrak{f} \triangleleft \mathcal{O}_K$ a *conductor ideal* if there exists some order $\mathcal{O}$ in K whose conductor equals $\mathfrak{f}$.

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It is well known that not every ideal of $\mathcal{O}_K$ is a conductor ideal, and P. Furtwängler [2] gave a characterization of all conductor ideals, which probably was already known to Dedekind (see [3], p. 548). A reproof of Furtwängler's result in modern terminology was recently given in [5], and generalized in [9]. It is also well known that for a characterization of all possible orders of a number field it suffices to characterize all orders whose conductor is a given ideal $\mathfrak{f}$ with $\mathcal{N}(\mathfrak{f})$ a power of a rational prime (see e.g. Th. 1.1 in [5]).

In his PhD thesis [8] the second author determined all possible orders in pure cubic number fields. It turned out that his result was independent of any arithmetic properties of the number fields, and in this paper we generalize the result of [8] to any cubic number field.

Theorem 1. *Let K be a cubic number field, $p \in \mathbb{P}$ a rational prime, $\mathfrak{f} \lhd \mathcal{O}_K$ a conductor ideal whose norm is a power of p and put $\mathcal{O} = \mathbb{Z} + \mathfrak{f}$, the minimal order with conductor $\mathfrak{f}$.*

Let $d \in \mathbb{N}_0$ be maximal with $\mathfrak{f} \subset (p^d)$, put $d_1 = \left[\frac{d}{2}\right]$ and let $e \in \{0, 1, 2, 3\}$ denote the number of different conductor ideals $\mathfrak{f}' \supset \mathfrak{f}$ such that $(\mathbb{Z} + \mathfrak{f}')/\mathcal{O}$ is a group of order p.

a) *For any additive subgroup A of $\mathcal{O}_K$ with $\mathcal{O} \subset A \subset \mathcal{O}_K$ we have: A is an order with conductor $\mathfrak{f}$ if and only if $A/\mathcal{O}$ is cyclic of order p^k with $k \le \frac{d}{2}$ and A does not contain $\mathbb{Z} + \mathfrak{f}'$ for any conductor ideal $\mathfrak{f}'$ with $(\mathbb{Z} + \mathfrak{f}')/\mathcal{O}$ cyclic of order p.*

b) *In K there exist exactly*

$$p^{d_1} + (2 - e)\,\frac{p^{d_1} - 1}{p - 1}$$

different orders with conductor $\mathfrak{f}$.

From Theorem 1.a) we see that one can obtain all orders with conductor $\mathfrak{f}$, as soon as one has a basis for the group $\mathcal{O}_K/\mathcal{O}$. To obtain all orders with arbitrary conductor one has to apply the Chinese remainder theorem (see [5], Th. 1.1).

For any number field K, define the Dirichlet series counting the orders of K with respect to their index by

$$\eta_K(s) = \sum_{\mathcal{O} \text{ an order in } K} \frac{1}{(\mathcal{O}_K : \mathcal{O})^s} \, .$$

Up to an obvious factor, this equals the zeta function of orders of K (see [4]). J. Nakagawa used results of B. Datskovsky and D.J. Wright [1] to show that for cubic number fields K one has

$$\eta_K(s) = \frac{\zeta_K(s)}{\zeta_K(2s)}\,\zeta(2s)\zeta(3s-1) \, , \tag{1}$$

where $\zeta_K(s)$ and $\zeta(s)$ denote the zeta function of K and Riemann's zeta function, resp. (Lemma 3.2 in [7]).