Journal of Number Theory 166 (2016) 424–451

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

New normality constructions for continued fraction expansions

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A R T I C L E   I N F O

Article history:
Received 15 July 2015
Received in revised form 19 January 2016
Accepted 21 January 2016
Available online 30 March 2016
Communicated by David Goss

MSC:
11K50
11K16

Keywords:
Continued fractions
Normal numbers

A B S T R A C T

Text. Adler, Keane, and Smorodinsky showed that if one concatenates the finite continued fraction expansions of the sequence of rationals

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \dots$$

into an infinite continued fraction expansion, then this new number is normal with respect to the continued fraction expansion. We show a variety of new constructions of continued fraction normal numbers, including one generated by the subsequence of rationals with prime numerators and denominators:

$$\frac{2}{3}, \frac{2}{5}, \frac{3}{5}, \frac{2}{7}, \frac{3}{7}, \frac{5}{7}, \dots .$$

Video. For a video summary of this paper, please visit https://youtu.be/L7uyAQ7hS74.

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http://dx.doi.org/10.1016/j.jnt.2016.01.030

1. Introduction

A number $x \in [0, 1)$ is said to be normal (to base 10) if for any string $s = [d_1, d_2, \dots, d_k]$ of decimal digits, we have

$$\lim_{N\to\infty} \frac{A_s(N; x)}{N} = \frac{1}{10^k},$$

where $A_s(N; x)$ is the number of times the string s appears starting in the first N digits of the decimal expansion of x. For numbers outside of the interval $[0, 1)$, we consider them to be normal if the number taken modulo 1 is normal. While it is a simple consequence of the pointwise ergodic theorem that almost all real numbers are normal, there is no commonly used irrational number, such as π, e, or even $\sqrt{2}$, that is known to be normal.

However, mathematicians have constructed a wide variety of normal numbers, the first of which was found by Champernowne: he showed that the number

$$0.123456789101112131415\dots,$$

formed by concatenating all the natural numbers in order, is normal [4]. Following Champernowne, Besicovitch showed that the number

$$0.149162536496481100\dots,$$

formed by taking all the perfect squares in order, is normal [2]. These constructions inspired a large area of research, as mathematicians considered for which functions $f(n)$ would the number

$$0.f(1)f(2)f(3)\dots$$

be normal. A related question asks whether just concatenating the prime values of a function,

$$0.f(2)f(3)f(5)f(7)f(11)\dots,$$

also generates a normal number. A small selection of all the results in this area include the work of Davenport and Erdős [7]; Nakai and Shiokawa [14]; De Koninck and Kátai [8]; Madritsch, Thuswaldner, and Tichy [13]; and the author [20].

Of particular interest to this paper is the work of Copeland and Erdős [5]. They showed that almost all integers are (ϵ, k)-normal, which refers to the fact that each string of length k appears in the decimal expansion of the integer to within ϵ of the expected frequency 10^{-k}. Thus, in place of the sequence of all positive integers, as in Champernowne, if we take a sufficiently dense subset of the positive integers and concatenate those, we expect to get a number that is normal as well. Copeland and Erdős