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An application of the Hardy–Littlewood conjecture

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A B S T R A C T

We assume a weak Hardy–Littlewood conjecture and derive an upper bound for a real exceptional zero associated with a prime modulus.

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and in all of mathematics. It states that every even integer greater than 2 can be expressed as the sum of two primes.

In 1923 Hardy and Littlewood conjectured that

$$\sum_{\substack{p_1, p_2 \le N \\ p_1 + p_2 = N}} 1 \sim \frac{N}{\phi(N)} \prod_{p \dagger N} \left(1 - \frac{1}{(p-1)^2}\right) \frac{N}{\log^2 N}$$

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where N is an even integer and where p_1 and p_2 denote prime numbers and ϕ is Euler's phi-function.

Under a weaker assumption we prove an upper bound for a real exceptional zero associated with a prime modulus. Such a bound is well-known to have important applications, see [1] and [2].

Weak Hardy–Littlewood conjecture. There is an absolute constant $\delta > 0$ such that for all even integers $N > 2$,

$$\sum_{p_1+p_2=N} 1 \geq \frac{\delta N}{\log^2 N}$$

Theorem 1. *Let q be a prime number congruent to 3 modulo 4 and let χ be the real primitive character of modulus q. Suppose that the Dirichlet L-function $L(s,\chi)$ has a real exceptional zero β. If the Weak Hardy–Littlewood Conjecture is correct then there is a number $c > 0$ such that for any such q and β, we have*

$$\beta \leq 1 - \frac{c}{\log^2 q}$$

1. Some lemmas

The following lemmas are well-known and are easily found in the literature in references [1] and [2]

Lemma 1. *Let $e(x) = e^{2\pi i x}$. Then*

$$\sum_{k=1}^{m} e\left(\frac{kn}{m}\right) = \begin{cases} m & \text{if } n \equiv 0 \pmod m \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2 *(Prime number theorem). There is a constant $c_1 > 0$ such that*

$$\pi(x) = Li(x) + O\left(x \exp(-c_1\sqrt{\log x})\right)$$

for $x > 2$ where $Li(x) = \int_2^x \frac{du}{\log u} = \frac{x}{\log x} + O(\frac{x}{\log^2 x})$.

Lemma 3 *(Prime number theorem for arithmetic progressions). For any constant $C > 0$ there is a constant $c_2 > 0$ such that, uniformly for $q \leq \exp(C\sqrt{\log x})$, we have*

$$\pi(x;q,a) = \frac{Lix}{\phi(q)} - \frac{\chi(a)}{\phi(q)} \int_2^x \frac{u^{\beta-1}}{\log u} du + O\left(x \exp(-c_2\sqrt{\log x})\right)$$

here the term with χ mod q is only present if there exists a real zero $\beta > 1 - \frac{c}{\log q}$ of $L(s,\chi)$ (where $c > 0$ is a fixed positive constant independent of q).