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A supercongruence involving Delannoy numbers and Schröder numbers

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ABSTRACT

The Delannoy numbers and Schröder numbers are given by

$$D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \quad \text{and}$$

$$S_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{1}{k+1},$$

respectively. Let $p > 3$ be a prime. We mainly prove that

$$\sum_{k=1}^{p-1} D_k S_k \equiv 2p^3 B_{p-3} - 2pH^*_{p-1} \pmod{p^4},$$

where B_n is the n-th Bernoulli number and these H^*_n are the alternating harmonic numbers given by $H^*_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}$. This supercongruence was originally conjectured by Z.-W. Sun in 2011.

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1. Introduction

In combinatorics, the n-th Delannoy number describes the number of paths from $(0,0)$ to (n,n), using only steps $(1,0),(0,1)$ and $(1,1)$, while n-th Schröder number represents the number of such paths that do not rise above the line $y=x$. It is known that

$$D_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\quad\text{and}\quad S_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}.$$

Z.-W. Sun [8,9] have proved some amazing arithmetic properties of Delannoy numbers and Schröder numbers. For example, he showed that for any odd prime p,

$$\sum_{k=1}^{p-1}\frac{D_k}{k^2}\equiv 2\left(\frac{-1}{p}\right)E_{p-3}\pmod{p},$$

$$\sum_{k=0}^{p-1}D_k^2\equiv\left(\frac{2}{p}\right)\pmod{p},$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol and E_n is the n-th Euler number. In 2011, Z.-W. Sun [8] also raised some interesting conjectures involving these numbers, one of which was

Conjecture 1.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}D_kS_k\equiv -2p\sum_{k=1}^{p-1}\frac{(-1)^k+3}{k}\pmod{p^4}.\tag{1.1}$$

Define the following two polynomials:

$$D_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k\quad\text{and}\quad S_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{x^k}{k+1}.$$

Note that $D_n(\frac{x-1}{2})$ coincides with the Legendre polynomial $P_n(x)$. Recently, Guo [4] proved that for any prime p and integer x satisfying $p\nmid x(x+1)$,

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^3\equiv p\left(\frac{-4x-3}{p}\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^4\equiv p\pmod{p^2},$$

and thus confirmed some conjectures of Z.-W. Sun [9].