Journal of Number Theory 168 (2016) 247–256

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Digitally delicate primes

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A R T I C L E I N F O

Article history:
Received 14 October 2015
Received in revised form 18 April 2016
Accepted 19 April 2016
Available online 1 June 2016
Communicated by K. Soundararajan

Keywords:
Prime numbers
Covering congruences
Sieve methods

A B S T R A C T

Tao has shown that in any fixed base, a positive proportion of prime numbers cannot have any digit changed and remain prime. In other words, most primes are "digitally delicate". We strengthen this result in a manner suggested by Tao: A positive proportion of primes become composite under any change of a single digit and any insertion of a fixed number of arbitrary digits at the beginning or end.

1. Introduction

In a short note published in 2008, Tao [11] proved the following theorem:

Theorem 1.1. *Let $K \geq 2$ be an integer. For all sufficiently large integers N, the number of primes p between N and $(1+1/K)N$ such that $|kp + ja^i|$ is composite for all integers $1 \leq a, |j|, k \leq K$ and $0 \leq i \leq K \log N$ is at least $c_K \frac{N}{\log N}$ for some constant $c_K > 0$ depending on only K.*

The following consequence is immediate, in view of the prime number theorem (or Chebyshev's weaker estimates).

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http://dx.doi.org/10.1016/j.jnt.2016.04.007
0022-314X/

Corollary 1.2. *Fix a base $a \geq 2$. A positive proportion of prime numbers become composite if any single digit in their base a expansion is altered.*

The infinitude of the primes appearing in Corollary 1.2 had earlier been shown by Erdős [2]. (He assumes $a = 10$ but the argument generalizes in an obvious way.) When $a = 10$, these “digitally delicate” primes are tabulated as sequence A050249 in the OEIS, where they are called “weakly prime”.

At the conclusion of [11], Tao suggests a few ways his result could possibly be improved. In this paper we establish one of the suggested generalizations:

Theorem 1.3. *Fix an integer $K \geq 2$. There is a constant $c_K > 0$ such that the following holds for all sufficiently large N: Let $\mathcal{S}_N \subseteq [-KN, KN]$ be an arbitrary set of integers of cardinality at most K. Let K_N be the number of primes $N \leq p \leq (1+1/K)N$ such that $|kp + ja^i + s|$ is either equal to p or composite for all combinations of integers a, i, j, k, and s where $1 \leq a, |j|, k \leq K$, $0 \leq i \leq K \log N$, and $s \in \mathcal{S}_N$. Then $K_N \geq c_K \frac{N}{\log N}$.*

This immediately yields the following strengthening of Corollary 1.2.

Corollary 1.4. *In any fixed base, a positive proportion of prime numbers become composite if one modifies any single digit and appends a bounded number of digits at the beginning or end.*

As in Tao's work, the key idea of the proof is to use a partial covering along with an upper bound sieve. The following well-known estimate plays a critical role (see [5, Theorem 2.2, p. 68], [11, Corollary A.2]).

Lemma 1.5 *(Brun/Selberg upper bound). Let W and b be positive integers and let k and h be non-zero integers. If x is sufficiently large (depending on W and b), the number of primes $m \leq x$ where $m \equiv b \pmod W$ and $|km + h|$ is also prime is*

$$\ll_k \frac{x}{W(\log x)^2}\left(\prod_{p \mid W}\left(1-\frac{1}{p}\right)^{-2}\right)\left(\prod_{\substack{p \mid h \\ p \nmid W}}\left(1-\frac{1}{p}\right)^{-1}\right),$$

where the products are restricted to prime numbers p.

Whenever Lemma 1.5 is applied in Tao's proof of Theorem 1.1, the product over p dividing h is uniformly bounded. However, to prove Theorem 1.3, we must deal with cases where that product can be very large. To work around this, we show that such cases arise very rarely, so rarely that this product is bounded in a suitable average sense. To establish this, we need to invoke a classical theorem of Romanoff [8] about multiplicative orders, which originally appeared in his work on numbers of the form $p + 2^k$. (Actually we use a slightly strengthened form of Romanoff's result due to Erdős [1].)