Journal of Number Theory 168 (2016) 306–333

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

The level 13 analogue of the Rogers–Ramanujan continued fraction and its modularity ☆

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A R T I C L E   I N F O

Article history:
Received 27 November 2015
Received in revised form 16 April 2016
Accepted 16 April 2016
Available online 2 June 2016
Communicated by David Goss

MSC:
11J70
11F11
11R04
14H55

Keywords:
Rogers–Ramanujan continued fraction
Modular function
Modular unit

A B S T R A C T

We prove the modularity of the level 13 analogue $r_{13}(\tau)$ of the Rogers–Ramanujan continued fraction. We establish some properties of $r_{13}(\tau)$ using the modular function theory. We first prove that $r_{13}(\tau)$ is a generator of the function field on $\Gamma_0(13)$. We then find modular equations of $r_{13}(\tau)$ of level n for every positive integer n by using affine models of modular curves; this is an extension of Cooper and Ye's results with levels $n = 2, 3$ and 7 to every level n. We further show that the value $r_{13}(\tau)$ is an algebraic unit for any $\tau \in K - \mathbb{Q}$, where K is an imaginary quadratic field.

☆ The authors are supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2009-0093827), and the first author is also supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (2014-002731).

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http://dx.doi.org/10.1016/j.jnt.2016.04.009

1. Introduction

Let $\mathfrak{H}$ be the complex upper half plane and $q := e^{2\pi i\tau}$ for $\tau \in \mathfrak{H}$. The *Rogers–Ramanujan continued fraction* $r(\tau)$ is defined by

$$r(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}.$$

It can be also written as an infinite product as follows:

$$r(\tau) = q^{\frac{1}{5}} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{n}{5}\right)} \tag{1.1}$$

with the Jacobi symbol $\left(\frac{n}{N}\right)$. On the other hand, it has the following interesting properties:

$$\frac{1}{r(\tau)} - 1 - r(\tau) = q^{-\frac{1}{5}} \prod_{n=1}^{\infty} \frac{(1 - q^{\frac{n}{5}})}{(1 - q^{5n})} \tag{1.2}$$

and

$$\frac{1}{r^5(\tau)} - 11 - r^5(\tau) = q^{-1} \prod_{n=1}^{\infty} \frac{(1 - q^n)^6}{(1 - q^{5n})^6}. \tag{1.3}$$

The identity in (1.1) was proved in [9], and (1.2) and (1.3) were stated by Ramanujan [1, p. 85 and p. 267] and proved by Watson [11].

Recently, Gee and Honesbeek studied the modularity of $r(\tau)$ and evaluated $r(\tau)$ for an imaginary quadratic quantity τ [6]. Cais and Conrad investigated the modular equation of $r(\tau)$ and its properties in the view of arithmetic models of modular curves [3].

For any positive integer $N > 2$ with $\left(\frac{-1}{N}\right) = 1$, we define the *level N analogue of the Rogers–Ramanujan continued fraction* to be

$$r_N(\tau) := q^{\alpha_N} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{n}{N}\right)},$$

where

$$\alpha_N = \sum_{r=1}^{\left[\frac{N}{2}\right]} \frac{r(r-N)}{2N} \left(\frac{r}{N}\right)$$