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# Bivariate polynomial mappings associated with simple complex Lie algebras

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ABSTRACT

There are three families of bivariate polynomial maps associated with the rank-2 simple complex Lie algebras $A_2, B_2 \cong C_2$ and $G_2$. It is known that the bivariate polynomial map associated with $A_2$ induces a permutation of $\mathbf{F}_q^2$ if and only if $\gcd(k, q^s - 1) = 1$ for $s = 1, 2, 3$. In this paper, we give similar criteria for the other two families. As an application, a counterexample is given to a conjecture posed by Lidl and Wells about the generalized Schur's problem.

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## 0. Introduction

A polynomial map $f : \mathbf{C}^n \to \mathbf{C}^n$ of degree greater than one is called *integrable* if there exists a polynomial map $g : \mathbf{C}^n \to \mathbf{C}^n$ of degree greater than one such that $f$ and $g$ commute, i.e. $f \circ g = g \circ f$, and the set of iterations of $f$ and $g$ are disjoint. Integrable maps

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play an important role in the theory of dynamical systems because they show an unusual degree of symmetry [Ve91]. In the case $n = 1$, a full description of integrable polynomials was given by Julia [Ju22], Fatou [Fa24] and Ritt [Ri23]. An integrable polynomial map $f : \mathbf{C} \to \mathbf{C}$ can be transformed by a linear change of variables to the form $f = z^n$ or $f = \pm T_n(z)$, where $T_n(z) = \cos(n \arccos z)$ is the Chebyshev polynomial.

There is a question in the theory of finite fields which has a similar answer. A polynomial $f(x) \in \mathbf{Z}[x]$ is called *exceptional* if $f(x)$ induces a permutation of an infinite number of finite fields $\mathbf{F}_p$ where $p$ is prime. It is well known that a polynomial is exceptional if and only if it is a composition of linear polynomials, power maps and the Chebyshev polynomials. One side of this statement is relatively easier to prove since the $k$th power map and the $k$th Chebyshev polynomial induce a permutation of $\mathbf{F}_q$ if and only if $\gcd(k, q-1) = 1$ and $\gcd(k, q^2-1) = 1$, respectively [LN83]. The other side of this classification is known as Schur's problem and proved by Fried [Fr70]. Other proofs have been given by Turnwald [Tu95] and Müller [Mü97].

Let $\mathbf{P}^1(\mathbf{C})$ be the projective space of dimension one. Apart from the power maps and Chebyshev polynomials, there is one more family of rational maps on $\mathbf{P}^1(\mathbf{C})$ which satisfies the commuting relation $f \circ g = g \circ f$ [Ri23]. It is the family of Lattès maps induced by isogenies of an elliptic curve $E$. In our previous work [Kü14], using the underlying elliptic curve group structure, we gave a criterion when a Lattès map induces a permutation of $\mathbf{P}^1(\mathbf{F}_q)$. In the theory of dynamical systems, especially in its arithmetical aspects, the underlying algebraic structure plays an important role. For example, see Silverman [Si07].

In this paper, we pay attention to the bivariate polynomial mappings associated with the rank-2 simple complex Lie algebras. We fix some notation first. Let $\mathfrak{g}$ be a complex Lie algebra of rank $n$ and $\mathfrak{h}$ its Cartan subalgebra, $\mathfrak{h}^*$ its dual space, $\mathcal{L}$ a lattice of weights in $\mathfrak{h}^*$ generated by the fundamental weights $\omega_1, \ldots, \omega_n$, and $L$ the dual lattice in $\mathfrak{h}$. Veselov defines the mapping $\Phi_{\mathfrak{g}} : \mathfrak{h}/L \to \mathbf{C}^n$, $\Phi_{\mathfrak{g}}(\varphi_1, \ldots, \varphi_n)$,

$$\varphi_k = \sum_{w \in W} e^{2\pi i w(\omega_k)}$$

where $W$ is the Weyl group, acting on the space $\mathfrak{h}^*$. Veselov shows that there exist a family of polynomial mappings associated with each simple complex Lie algebra with nice dynamical properties. Hofmann and Withers give the same result independently somewhat later.

**Theorem 0.1** *([Ve87,HW88]). With each simple complex Lie algebra of rank $n$, there is an associated infinite series of integrable polynomial mappings $P^k_{\mathfrak{g}}$, determined from the conditions*

$$\Phi_{\mathfrak{g}}(k\mathbf{x}) = P^k_{\mathfrak{g}}(\Phi_{\mathfrak{g}}(\mathbf{x})).$$

*All coefficients of the polynomials defining $P^k_{\mathfrak{g}}$ are integers.*

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