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Multiplicative groups of fields and hereditarily irreducible polynomials ☆

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A B S T R A C T

In this paper we explore the concept of *good heredity* for fields from a group theoretic perspective. Extending results from [8], we show that several natural families of fields are of good heredity, and some others are not. We also construct several examples to show that various wishful thinking expectations are not true.

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1. Introduction

In this paper we investigate a few properties of fields closely related to the freeness of their multiplicative groups. The first author stumbled upon one of them in her investigation of divisibility of quasiendomorphisms of abelian varieties in [8]. This property of a field F, which we call *good heredity*, deals with (ir)reducibility of $P(x^n)$ for a polynomial $P(x)$ over F, as n varies. The other two are properties of the multiplicative group of the field, *probably* slightly weaker than freeness. We were surprised to discover that the two are not equivalent to each other, nor to good heredity (see Examples 17 and 22 and Section 7.2). One advantage of our good heredity over all these related properties of the multiplicative group is that good heredity passes to finite extensions of fields (see Theorem 30), while the various properties of the multiplicative group do not (see example in §7.3 and §7.1).

Definition 1. For a field F,

- let $F^\times$ denote the multiplicative group of F, an abelian group;
- let $\mu(F)$ denote the group of roots of unity on F, i.e. the torsion subgroup of $F^\times$;
- and let $F^\times/\mu(F)$ denote their quotient, a torsion-free abelian group.

May [5] shows that any locally cyclic abelian group can show up as a direct summand of $F^\times$. May [7] proves that for many interesting fields, $F^\times/\mu(F)$ is a free abelian group; and constructs an example showing that this property (freeness of $F^\times/\mu(F)$) does not pass to finite extensions, even in the very tame setting of algebraic extensions of $\mathbb{Q}$.

Definition 2. An abelian group G is *rootless* if for any non-torsion element $a \in G$, the divisible hull inside G of the subgroup of G generated by a is free. We call G *free modulo torsion* if the quotient of G by its torsion subgroup is free Abelian.

A field F is *rootless* if the group $F^\times$ is rootless.

A field F is *rootless modulo torsion* if the group $F^\times/\mu(F)$ is rootless.