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Analytic properties of multiple zeta functions and certain weighted variants, an elementary approach

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In this article we obtain the meromorphic continuation of multiple zeta functions, together with a complete list of their poles and residues, by means of an elementary and simple *translation formula* for these multiple zeta functions. The use of matrices to express this translation formula leads, in particular, to a succinct description of the residues of the multiple zeta functions. We conclude our paper by introducing certain interesting weighted variants of multiple zeta functions. They are shown to behave particularly nicely with respect to product formulas and location of poles.

1. Introduction

When $r \geq 1$ is an integer, let U_r denote the subset of $\mathbb{C}^r$ consisting of the r-tuples $(s_1, \dots, s_r)$ of complex numbers satisfying the conditions

$$\mathrm{Re}(s_1 + \cdots + s_i) > i \qquad \text{for } 1 \leq i \leq r. \tag{1}$$

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Then U_r is a convex open subset of $\mathbb{C}^r$ and the *multiple zeta function of depth* r is the complex valued function on U_r defined by

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > \cdots > n_r > 0} n_1^{-s_1} \cdots n_r^{-s_r}. \tag{2}$$

The series on the right hand side of (2) is normally convergent on any compact subset of U_r (see Section 2). Consequently, $(s_1, \ldots, s_r) \mapsto \zeta(s_1, \ldots, s_r)$ is a holomorphic function on U_r. When $r = 1$ this function is the same as the classical Riemann zeta function defined by the Dirichlet series $\sum_{n>0} n^{-s}$ for $\mathrm{Re}(s) > 1$. It is convenient to extend the notion of a multiple zeta function to $r = 0$ as well: in this case U_0 consists only of the empty tuple, denoted by $\varnothing$, and we set $\zeta(\varnothing) = 1$.

For any $r \geq 0$, the multiple zeta function of depth r can be extended meromorphically to $\mathbb{C}^r$. While this may be deduced from the general works [9,10] and [5], a number of authors have found it useful to present direct proofs of this result in the context of multiple zeta functions. For instance, J. Zhao ([15], Theorem 5) proves it with the aid of the theory of generalized functions, and A. Goncharov, in an unpublished preprint, obtains it by means of wave front sets of distributions (see [7], Theorem 2.25). A thorough description of the meromorphic continuation of the more general Hurwitz multiple zeta functions has been given by J.P. Kelliher and R. Masri in [8] using Mellin transformation of meromorphic distributions. A simpler approach to this subject has been proposed by Akiyama, Egami and Tanigawa ([2], Theorem 1), where the required meromorphic continuation was obtained by applying the classical Euler–Maclaurin formula to the first index of the summation n_1 in (2). An alternate proof using Mellin–Barnes integrals has been given by K. Matsumoto ([11], Proposition 6).

In the first part of this paper we begin by presenting what appears to be the simplest and most direct proof of the meromorphic continuation of multiple zeta functions, to the best of our knowledge. In effect, developing on a remark of Jean Ecalle in [3] and [4], we show that the existence of the meromorphic continuation of multiple zeta functions can be immediately read off from the following easily established identity.

Theorem 1. *For any integer $r \geq 2$ and all $(s_1, \ldots, s_r)$ in the open set U_r, we have*

$$\zeta(s_1 + s_2 - 1, s_3, \ldots, s_r) = \sum_{k \geq 0} \frac{(s_1 - 1)s_1 \cdots (s_1 + k - 1)}{(k+1)!} \zeta(s_1 + k, s_2, \ldots, s_r), \tag{3}$$

where the series on the right hand side converges normally on any compact subset of U_r.

We call (3) the *translation formula* for the multiple zeta function of depth $r \geq 2$. When $r = 1$, that is, in the case of Riemann zeta function, it has long been known that we have a similar formula

$$1 = \sum_{k \geq 0} \frac{(s-1) \cdots (s+k-1)}{(k+1)!} (\zeta(s+k) - 1), \tag{4}$$