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On a divisibility relation for Lucas sequences

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A B S T R A C T

In this note, we study the divisibility relation $U_m \mid U_{n+k}^s - U_n^s$, where $\mathbf{U} := \{U_n\}_{n\geq 0}$ is the Lucas sequence of characteristic polynomial $x^2 - ax \pm 1$ and k, m, n, s are positive integers.

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1. Introduction

Let $\mathbf{U} := \mathbf{U}(a,b) = \{U_n\}_{n\ge 0}$ be the Lucas sequence given by $U_0 = 0$, $U_1 = 1$ and

$$U_{n+2} = aU_{n+1} + bU_n \quad \text{for all} \quad n \ge 0, \quad \text{where} \quad b \in \{\pm 1\}. \tag{1}$$

Its characteristic equation is $x^2 - ax - b = 0$ with roots

$$(\alpha, \beta) = \left(\frac{a + \sqrt{a^2+4b}}{2}, \frac{a - \sqrt{a^2+4b}}{2} \right). \tag{2}$$

When $a \ge 1$, we have that $\alpha > 1 > |\beta|$. We assume that $\Delta = a^2 + 4b > 0$ and that α/β is not a root of unity. This only excludes the pairs $(a,b) \in \{(0,\pm 1),\ (\pm 1,-1),\ (2,-1)\}$ from the subsequent considerations. Here, we look at the relation

$$U_m \mid U_{n+k}^s - U_n^s, \tag{3}$$

with positive integers k, m, n, s. Note that when $(a,b) = (1,1)$, then $U_n = F_n$ is the nth Fibonacci number. Taking $k = 1$ and using the relations

$$\begin{aligned} F_{n+1} - F_n &= F_{n-1}, \\ F_{n+1} + F_n &= F_{n+2}, \\ F_{n+1}^2 + F_n^2 &= F_{2n+1}, \end{aligned}$$

it follows that relation (3) holds with $s = 1, 2, 4$, and $m = n-1,\ n+1,\ 2n+1$, respectively. Further, in [2], the authors assumed that m and n are coprime positive integers. In this case, F_n and F_m are coprime, so the rational number F_{n+1}/F_n is defined modulo F_m. Then it was shown in [2] that if this last congruence class above has multiplicative order s modulo F_m and $s \notin \{1,2,4\}$, then

$$m < 500s^2. \tag{4}$$

In this paper, we study the general divisibility relation (3) and prove the following result.

Theorem 1. *Let a be a non-zero integer, $b \in \{\pm 1\}$, and k a positive integer. Assume that $(a,b) \notin \{(\pm 1,-1),\ (\pm 2,-1)\}$. Given a positive integer m, let s be the smallest positive integer such that divisibility (3) holds. Then either $s \in \{1,2,4\}$, or*

$$m < 20000(sk)^2. \tag{5}$$