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Joint universality and generalized strong recurrence with rational parameter

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ABSTRACT

We prove that, for every rational $d \neq 0, \pm 1$ and every compact set $K \subset \{s \in \mathbb{C} : 1/2 < \mathrm{Re}(s) < 1\}$ with connected complement, any analytic non-vanishing functions f_1, f_2 on K can be approximated, uniformly on K, by the shifts $\zeta(s + i\tau)$ and $\zeta(s + id\tau)$, respectively. As a consequence we deduce that the set of τ satisfying $|\zeta(s + i\tau) - \zeta(s + id\tau)| < \varepsilon$ uniformly on K has a positive lower density for every $d \neq 0$.

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1. Introduction

In 1981 Bagchi [1] discovered an interesting connection between the Riemann Hypothesis and Voronin's universality theorem (see [18]) for the Riemann zeta function $\zeta(s)$. Namely, he proved that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > \frac{1}{2}$ if and only if for every compact set $K \subset \{s \in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(s) < 1\}$ with connected complement and every $\varepsilon > 0$ we have

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] : \max_{s\in K} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon \right\} > 0, \tag{1}$$

where meas$\{\cdot\}$ denotes the real Lebesgue measure. In the language of topological dynamics (see [6]) (1) is called the *strong recurrence* property for the Riemann zeta function.

Bagchi's observation was extended to the case of Dirichlet L-functions by himself in [2] and [3], and to the case of general universal L-functions, for which the Generalized Riemann Hypothesis is expected, in [17, Theorem 8.4].

Nakamura [10] suggested the following related problem: find all d such that for every compact set $K \subset \{s \in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(s) < 1\}$ with connected complement and every $\varepsilon > 0$ we have

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] : \max_{s\in K} |\zeta(s+i\tau) - \zeta(s+id\tau)| < \varepsilon \right\} > 0. \tag{2}$$

This property can be called *generalized strong recurrence* with parameter d. However, it should be noted that sometimes in the literature it is called also the self-approximation property with parameter d. Using this notion Bagchi's result states that the Riemann Hypothesis is equivalent to the generalized strong recurrence property for $\zeta(s)$ with parameter $d = 0$.

Nakamura, in the same paper, gave the partial answer to this question by proving that (2) holds if d is algebraic irrational. He also observed that the generalized strong recurrence property holds for almost all real parameters d. His result was improved by the author in [13] to all irrational parameter d. The positive answer for non-zero rational d was claimed by Garunkštis [5] and Nakamura [11]. Unfortunately, their arguments have a gap, which was pointed out by Nakamura and Pańkowski [12] and partially filled, in the same paper, for all non-zero rational $d = \frac{a}{b}$ with $\gcd(a,b) = 1$ and $|a-b| \neq 1$.

The crucial step in the proof of the generalized strong recurrence property with parameter d is to show that the following set

$$\{\log p : p \text{ is prime}\} \cup \{d \log p : p \text{ is prime}\} \tag{3}$$

is linearly independent over $\mathbb{Q}$. This was proved for all algebraic irrational d and for almost all d by Nakamura [10]. Moreover, by using the six exponential theorem from the theory of transcendental numbers, the author noticed in [13] that for a given irrational d only a finite number of primes p can possibly be involved in the linear dependence of (3). This allowed to prove the following joint universality theorem, which easily implies