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The limit of the zero set of polynomials of the Fibonacci type

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ABSTRACT

The paper concerns the limit of the zero set S_n of the Fibonacci-type polynomials $G_n(x) = G_n(a, b, x)$ given by $G_0(x) = a$, $G_1(x) = x + b$, and $G_{n+2}(x) = xG_{n+1}(x) + G_n(x)$. It presents an explicit set L consisting of at most two points such that S_n converges to the union of $[-2i, 2i]$ and L in the Hausdorff metric. The proof is based on the asymptotic analysis of the eigenvalues of special non-Hermitian perturbations of Hermitian Toeplitz matrices.

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1. Introduction

Let a, b be complex numbers. The Fibonacci-type polynomials $G_n(\lambda) = G_n(a, b, \lambda)$ are defined by $G_0(\lambda) = a$, $G_1(\lambda) = \lambda + b$, and $G_{n+2}(\lambda) = \lambda G_{n+1}(\lambda) + G_n(\lambda)$. The

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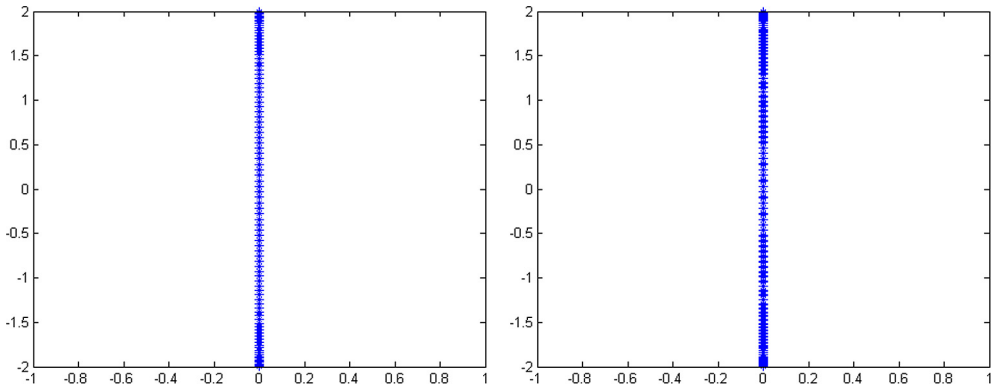

Fig. 1. The zeros of $G_{100}(1,0,\lambda)$ (left) and $G_{100}(2,0,\lambda)$ (right). There are indeed no outliers.

polynomials $G_n(1,0,\lambda)$, $G_n(2,0,\lambda)$, and $G_n(-1,-1,\lambda)$ are referred to as the Fibonacci, Lucas, and golden polynomials, respectively. We here prove the following.

Theorem 1.1. *Let a, b be complex numbers, let Δ be the set*

$$\Delta = \{\delta \in \mathbf{C} : |\delta| < 1, (a-1)\delta^2 + b\delta + 1 = 0\},$$

and let L be the set

$$L = \{\lambda \in \mathbf{C} \setminus [-2i, 2i] : \lambda = -b - a\delta \ \textit{ for some } \ \delta \in \Delta\}.$$

Then, as $n \to \infty$, the set of the zeros of $G_n(a,b,\lambda)$ converges in the Hausdorff metric to $[-2i, 2i] \cup L$.

Thus, eventually the zeros of $G_n(a,b,\lambda)$ cluster around the segment $[-2i, 2i]$ with the exception of at most two outliers, which cluster at the points of L. The proof will be given in Sections 2 and 3. Here are a few examples.

Example 1.2. Consider $G_n(a,0,\lambda)$. Then L is the empty set for $|a-1| \le 1$ and the doubleton consisting of the two points $\pm a/\sqrt{1-a}$ for $|a-1| > 1$. See Figs. 1 and 2. The zeros are plotted as asterisks, the points in L as circles.

Example 1.3. For the polynomials $G_n(a,a,\lambda)$, the roots of the polynomials giving the set Δ are -1 and $1/(1-a)$. Consequently, $L = \emptyset$ if $|a-1| \le 1$ and $L = \{(a^2-2a)/(a-1)\}$ if $|a-1| > 1$. See Fig. 3.

Example 1.4. We consider $G_n(a,1,\lambda)$. For $a = 0$, we have $\Delta = \{(1-\sqrt{5})/2\}$ and hence there is the single outlier $-1 - 0 \cdot (1-\sqrt{5})/2 = -1$. If $a = 2$, then both roots of $(a-1)\delta^2 + b\delta + 1 = \delta^2 + \delta + 1$ have modulus 1, and hence $L = \emptyset$. Figs. 4 and 5 illustrate

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