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An Euler totient sum inequality

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ABSTRACT

Text. Define $\chi(n)$ recursively by $\chi(1) = 1$ and $\chi(n) = \phi(n) + \chi(n/q)$ for all integers $n > 1$, where q is the least prime factor of n, and where ϕ is the Euler totient function. We show that $\chi(n) = \phi(d)(\chi(\ell) - 1) + \chi(d)$, where $n = d\ell$ and the prime factors of d are greater than the prime factors of ℓ. We also show $\chi(nm) \leq \chi(n)\chi(m)$ when n and m are coprime numbers. As an application, we show that for all primes $p \geq 11$, $\chi(p^2 - p) > \chi(p^2 - 1)$. We discuss the interpretation of χ as the clique number of the power graph of a finite cyclic group and the significance of the inequality in this context.

Video. For a video summary of this paper, please visit https://youtu.be/p8finzAEJps.

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1. Introduction

We discuss a problem of number theory that arose in connection with a question concerning the clique number of power graphs of finite groups. As is customary, *number* refers to positive integers.

Definition 1.1. Let G be a finite group. The *power graph* of G has vertex set G. Call distinct x, $y \in G$ *adjacent* if x is a power of y or if y is a power of x. By a *clique* of G, we mean a subset of mutually pairwise adjacent elements. By the *clique number* of G we mean the size $\chi(G)$ of the largest clique in G.

Directed power graphs were introduced in [8], and undirected power graphs–the type considered here–were first studied in [2]. See the survey [1] for more on power graphs. Clique numbers of power graphs were investigated in [6,10]. By [7, Corollary 2.5] and [6, Theorem 5], the clique number and chromatic number of a power graph coincide. Chromatic numbers of power graphs were studied in [3]. The references [10, Theorem 2] and [6, Theorem 7] give explicit formulas for the clique number of the cyclic group $\mathbb{Z}_n$ based upon the following.

Definition 1.2. Define $\chi : \mathbb{Z}^+ \to \mathbb{Z}^+$ recursively by $\chi(1) = 1$, and for all numbers $n > 1$, $\chi(n) = \phi(n) + \chi(n/q)$, where q is the least prime factor of n. We refer to $\chi(n)$ as the *clique number of* n.

Theorem 1.3. *(See [10, Theorem 2], [6, Theorem 7].) For all numbers* n,

$$\chi(\mathbb{Z}_n) = \chi(n).$$

We derive some properties of χ using only elementary number theory which can be found in most textbooks on the subject.

Theorem 1.4. *Suppose* $n = d\ell$, *where the prime factors of* d *are greater than the prime factors of* ℓ. *Then*

$$\chi(n) = \phi(d)(\chi(\ell) - 1) + \chi(d).$$

Theorem 1.5. *Let* m *and* n *be coprime numbers. Then*

$$\chi(nm) \leq \chi(n)\chi(m),$$

with equality if and only if one of m *and* n *is 1.*

We apply these results to give an infinite family of counterexamples to the following.