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Arithmetic properties of the sequence of derangements

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A R T I C L E I N F O

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A B S T R A C T

The sequence of derangements is given by the formula $D_0 = 1$, $D_n = nD_{n-1} + (-1)^n$, $n > 0$. It is a classical object appearing in combinatorics and number theory. In this paper we consider such arithmetic properties of the sequence of derangements as: periodicity modulo d, where $d \in \mathbb{N}_+$, p-adic valuations and prime divisors. Next, we use them to establish arithmetic properties of the sequences of even and odd derangements.

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1. Introduction

By the term of *derangement* we call a permutation in S_n without fixed points. We define *the n-th number of derangements* as the number of all derangements of the set with n elements. We denote this number by D_n. The sequence $(D_n)_{n\in\mathbb{N}}$, can be described by the recurrence $D_0 = 1$, $D_n = nD_{n-1} + (-1)^n$, $n > 0$. The sequence $(D_n)_{n\in\mathbb{N}}$ is a subject of research of many mathematicians. It is connected to other well known sequences. In particular, the sequence of numbers of derangements (or shortly, the sequence of derangements) appears in a natural way in the paper [22], devoted to the Bell numbers.

In [17] we proved a criterion for behavior of p-adic valuation of the Schenker sum a_n, given by the formula $a_n = \sum_{j=0}^{n} \frac{n!}{j!} n^j$, $n \in \mathbb{N}$. We expected that the method of proving this criterion could be used to other class of integer sequences. The trial of generalization of this method is one of the motivations for preparing this paper. The main point of this paper is to investigate arithmetic properties of sequence of derangements.

In Section 2 we set conventions and recall facts which are used in further parts of the paper.

In Section 3 we define pseudo-polynomial decomposition modulo p of a given sequence. If a sequence has this property then we can use the same method of proof as in [17] to obtain the description of p-adic valuation of elements of this sequence. Furthermore, we show that a sequence with pseudo-polynomial decomposition modulo p can be expressed as a product of functions f and g, where $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a p-adic continuous function which can be approximated by polynomials with integer coefficients and $g : \mathbb{N} \to \mathbb{Z}_p \backslash p\mathbb{Z}_p$.

The results from Section 3 are used in Section 4 to study arithmetic properties of the sequence of derangements. More precisely, the Section 4.1 is concerned with the periodicity of the sequence $(D_n \pmod{d})_{n\in\mathbb{N}}$ of remainders modulo d of the sequence of derangements, where $d \in \mathbb{N}_+$. Due to the divisibility $n - 1 \mid D_n$ for all $n \in \mathbb{N}$, we study prime divisors and p-adic valuations of the numbers $\frac{D_n}{n-1}$ with $n \geq 2$. We prove that the set of prime divisors of the numbers $\frac{D_n}{n-1}$, $n \geq 2$, is infinite.

Section 4.2 is devoted to some recurrence relations for the sequence of derangements and the polynomials

$$f_d = \sum_{j=0}^{d-1} (-1)^j \prod_{i=0}^{j-1} (X - i) \in \mathbb{Z}[X],$$