Journal of Number Theory 163 (2016) 211–237

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Cyclotomic coefficients: gaps and jumps

Oana-Maria Camburu [a], Emil-Alexandru Ciolan [b,*], Florian Luca [c], Pieter Moree [d], Igor E. Shparlinski [e]

[a] *Department of Computer Science, University of Oxford, Oxford OX1 3QD, United Kingdom*
[b] *Rheinische Friedrich-Wilhelms-Universität Bonn, Regina-Pacis-Weg 3, D-53113 Bonn, Germany*
[c] *School of Mathematics, University of the Witwatersrand, Private Bag X3, Wits 2050, South Africa*
[d] *Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany*
[e] *Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia*

A R T I C L E   I N F O

Article history:
Received 31 July 2015
Received in revised form 25 November 2015
Accepted 25 November 2015
Available online 14 January 2016
Communicated by David Goss

MSC:
11B83
11L07
11N25

Keywords:
Coefficients of cyclotomic polynomials
Products of primes
Numerical semigroups
Double Kloosterman sums

A B S T R A C T

We improve several recent results by Hong, Lee, Lee and Park (2012) on gaps and Bzdęga (2014) on jumps amongst the coefficients of cyclotomic polynomials. Besides direct improvements, we also introduce several new techniques that have never been used in this area.

* Corresponding author.
E-mail addresses: oana-maria.camburu@cs.ox.ac.uk (O.-M. Camburu), ciolan@uni-bonn.de (E.-A. Ciolan), florian.luca@wits.ac.za (F. Luca), moree@mpim-bonn.mpg.de (P. Moree), igor.shparlinski@unsw.edu.au (I.E. Shparlinski).

http://dx.doi.org/10.1016/j.jnt.2015.11.020
0022-314X/

1. Introduction

As usual, for an integer $n \geq 1$, we use $\Phi_n(Z)$ to denote the *nth cyclotomic polynomial*, that is,

$$\Phi_n(Z) = \prod_{\substack{j=0 \\ \gcd(j,n)=1}}^{n-1} (Z - \mathbf{e}_n(j)),$$

where for an integer $m \geq 1$ and a real z, we put

$$\mathbf{e}_m(z) = \exp(2\pi i z/m).$$

Clearly $\deg \Phi_n = \varphi(n)$, where $\varphi(n)$ is the Euler function. Using the above definition one sees that

$$Z^n - 1 = \prod_{d|n} \Phi_d(Z). \tag{1.1}$$

The Möbius inversion formula then yields

$$\Phi_n(Z) = \prod_{d|n} (Z^d - 1)^{\mu(n/d)}, \tag{1.2}$$

where $\mu(n)$ denotes the Möbius function.

We write

$$\Phi_n(Z) = \sum_{k=0}^{\varphi(n)} a_n(k) Z^k.$$

For $n > 1$ clearly $Z^{\varphi(n)}\Phi_n(1/Z) = \Phi_n(Z)$ and so

$$a_n(k) = a_n(\varphi(n) - k), \qquad 0 \leq k \leq \varphi(n), \qquad n > 1. \tag{1.3}$$

Recently, there has been a burst of activity in studying the cyclotomic coefficients $a_n(k)$, see, for example, [3–6,10,15,17–19,31,35] and references therein. Furthermore, in several works *inverse cyclotomic polynomials*

$$\Psi_n(Z) = (Z^n - 1)/\Phi_n(Z)$$

have also been considered, see [7,21–23,29].

The identities $\Phi_{2n}(Z) = \Phi_n(-Z)$, with $n > 1$ odd and $\Phi_{pm}(Z) = \Phi_m(Z^p)$ if $p \mid m$, show that, as far as the study of coefficients is concerned, the complexity of $\Phi_n(Z)$ is determined by its number of distinct odd prime factors. Most of the recent activity concerns the so-called *binary* and *ternary* cyclotomic polynomials, which are polynomials