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Asymptotic expansions related to the Glaisher–Kinkelin constant and its analogues

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ABSTRACT

Based on the Bell polynomials, Chen and Lin (2013) [9] obtained explicitly the coefficients a_m $(m = 1, 2, \ldots)$ in the following asymptotic expansion related to the Glaisher–Kinkelin constant A:

$$1^1 2^2 \cdots n^n \sim A \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(\sum_{m=0}^{\infty} \frac{a_m}{n^m} \right)^{1/r}.$$

In this paper, we provide a recurrence formula for the computation of these coefficients, and further describe their asymptotic behavior. We also give recurrence formulas, explicit expressions and asymptotic formulas for the coefficients of similar approximation formulas for two analogues of the Glaisher–Kinkelin constant.

1. Introduction

In the theory of mathematical constants, the Glaisher–Kinkelin constant or Glaisher's constant, typically denoted by A, is usually defined by the limit

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$$A = \lim_{n\to\infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k \tag{1.1}$$

(see [17,18,26]), as well as

$$\frac{e^{1/12}}{A} = \lim_{n\to\infty} \frac{G(n+1)}{(2\pi)^{n/2} n^{n^2/2-1/12} e^{-3n^2/4}}, \tag{1.2}$$

where $G(n)$ is the Barnes G-function [8]. It is named after mathematicians James Whitbread Lee Glaisher and Hermann Kinkelin. Its approximate value is $A \approx 1.28242712\cdots$.

The Glaisher–Kinkelin constant has been applied in the area of special functions more and more widely. As pointed out in [9], the Glaisher–Kinkelin constant A appears in a number of sums and integrals, especially those involving gamma functions and zeta functions. Finch introduced this constant A in a section of his book [16].

Making use of the derivative of the Riemann zeta function denoted by $\zeta'(z)$ [13,14], the Glaisher–Kinkelin constant A can be expressed by the following closed forms [12]:

$$A = e^{\frac{1}{12}-\zeta'(-1)} = (2\pi)^{\frac{1}{12}} \left(e^{\gamma \frac{\pi^2}{6} - \zeta'(2)}\right)^{\frac{1}{2\pi^2}}, \tag{1.3}$$

where $\gamma = 0.57721566\cdots$ is the Euler–Mascheroni constant.

Besides the closed forms, many authors made great efforts in the area of establishing more precise inequalities and more accurate approximations for the Glaisher–Kinkelin constant, and also had a lot of inspiring results, see for example [7,9,8,11,21,22].

Recall that B_i is the ith Bernoulli number defined by the power series expansion

$$\frac{x}{e^x-1} = \sum_{i=0}^{\infty} B_i \frac{x^i}{i!} = 1 - \frac{x}{2} + \sum_{i=1}^{\infty} B_{2i} \frac{x^{2i}}{(2i)!}. \tag{1.4}$$

It is well known that $B_{2i+1} = 0$, for all $i \geq 1$, and the first few Bernoulli numbers are $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$ and $B_6 = 1/42$.

Recently, Chen and Lin [9] established the following asymptotic formula

$$1^1 2^2 \cdots n^n \sim A \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(\sum_{m=0}^{\infty} \frac{a_m}{n^m}\right)^{1/r}, \tag{1.5}$$

as $n \to \infty$, with the coefficients a_m $(m = 1, 2, \ldots)$ explicitly given by

$$a_m = \frac{1}{m!} Y_m \left(-\frac{r1!B_3}{1\cdot 2\cdot 3}, -\frac{r2!B_4}{2\cdot 3\cdot 4}, \ldots, -\frac{rm!B_{m+2}}{m\cdot(m+1)\cdot(m+2)}\right). \tag{1.6}$$

Their main results are based on the exponential complete Bell polynomials Y_n, which are important tools in enumerative combinatorics [1,15]. The exponential complete Bell polynomials $Y_n := Y_n(x_1, x_2, \ldots, x_n)$ are defined by