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The doubles of one half of a numerical semigroup

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Let $\mathcal{F}$ be the set of all numerical semigroups and d a positive integer. Define a mapping $f_d : \mathcal{F} \to \mathcal{F}$ for every numerical semigroup S by $f_d(S) = \frac{S}{d}$. In this paper, we show that the equivalence class $(S)_{kerf_d}$ contains only finite maximal elements, but does not contain minimal element for every integer $d \geq 2$. Moreover, we study the subset of $(S)_{kerf_2}$ whose elements contain S and the subset of $(S)_{kerf_2}$ whose elements are contained in S. In particular, we study the case of S has embedding dimension two.

1. Introduction and preliminaries

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* is a subset S of $\mathbb{N}$ such that it is closed under addition, $0 \in S$ and $\mathbb{N} \backslash S$ is finite. The elements of $G(S) = \mathbb{N} \backslash S$ are the *gaps* of S and its cardinality, denoted by $g(S)$, is called the *genus* of S. Another

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important invariant of S is the largest integer not belonging to S, known as the *Frobenius number* of S and denoted by $F(S)$.

If $A \subseteq \mathbb{N}$ is a nonempty set, we denote by $\langle A\rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, that is, $\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}\backslash\{0\},\ a_1, \cdots, a_n \in A,\ \lambda_1, \cdots, \lambda_n \in \mathbb{N}\}$. It is well known that $\langle A\rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$. If S is a numerical semigroup and $S = \langle A\rangle$, then we say that A is a *system of generators* of S. In addition, if no proper subset of A generates S, then we say that A is a *minimal system of generators* of S. In [7] it is proved that every numerical semigroup S has a unique minimal system of generators and, moreover, such a system is finite. Denote by $M(S)$ the minimal system of generators of S. Let $M(S) = \{n_1 < n_2 < \cdots < n_p\}$. Then n_1 is known as the *multiplicity* of S, denoted by $m(S)$. The cardinality of $M(S)$, p, is called the *embedding dimension* of S and is denoted by $e(S)$. In particular, if $e(S) = m(S)$ then S is a numerical semigroup with *maximal embedding dimension*.

A *partially ordered set* $(A, \leq)$ is a nonempty set A with an order relation $\leq$. For $a, b \in A$, if $a \leq b$ or $b \leq a$, then a and b are *comparable*. Otherwise they are *incomparable*. A partially ordered set $(A, \leq)$ is called a *totally ordered set* (also called a *chain*) if any two elements of A are comparable. An element $a \in A$ is *maximal* in A if for every $c \in A$ that is comparable to a, $c \leq a$. An element $b \in A$ is *minimal* in A if for every $c \in A$ that is comparable to b, $b \leq c$.

Let S be a numerical semigroup and d a positive integer. Then in [7] it is proved that $\frac{S}{d} = \{x \in \mathbb{N} \mid dx \in S\}$ is also a numerical semigroup, called the *quotient* of S by d. The work on the numerical semigroups that are quotients of numerical semigroups of maximal embedding dimension and the numerical semigroups whose quotients are of maximal embedding dimension can be seen in [3,8]. When $d = 2$ we say that $\frac{S}{2}$ is one *half* of the numerical semigroup S. Dually, we say that S is a *double* [4] of the numerical semigroup $\frac{S}{2}$. Denote by $D(S)$ the set of all doubles of the numerical semigroup S, that is, $D(S) = \{T \text{ numerical semigroup} \mid S = \frac{T}{2}\}$. In [4], A.M. Robles-Pérez et al. gave a representation for the elements of $D(S)$ by a unique pair (m, H), where m is an odd integer of S and H is an m-upper subset of $G(S)$. M. D'Anna and F. Strazzanti studied the set $D(S)$ by the numerical duplication in [3]. Recently, F. Strazzanti gave a formula for the minimal genus of a multiple of a numerical semigroup S, a formula for the minimal genus of a symmetric double of S and the formula for the Frobenius number of the quotient of some families of numerical semigroups in [9].

Let S and T be two numerical semigroups such that $S \subseteq T$, and let d be a positive integer. T is called a *d-extension* [6] of S if $dT = \{dx \mid x \in T\} \subset S$. In [6], J.C. Rosales and M.B. Branco characterized the numerical semigroup that is a 2-extension of $\langle n_1, n_2\rangle$ by a subset of incomparable elements of $B(\langle n_1, n_2\rangle)$, where $B(\langle n_1, n_2\rangle) = \{x \in G(\langle n_1, n_2\rangle) \mid 2x \in \langle n_1, n_2\rangle\}$.

In this paper, we want to study the set of numerical semigroups that have the same quotient by a positive integer. Let $\mathcal{F}$ be the set of all numerical semigroups and d a positive integer. Then $\mathcal{F}$ is a partially ordered set under the relation of set inclusion. We define a mapping $f_d : \mathcal{F} \to \mathcal{F}$ for every numerical semigroup S by $f_d(S) = \frac{S}{d}$.