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Limit theorems related to beta-expansion and continued fraction expansion

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ABSTRACT

Let $\beta > 1$ be a real number and $x \in [0, 1)$ be an irrational number. Denote by $k_n(x)$ the exact number of partial quotients in the continued fraction expansion of x given by the first n digits in the β-expansion of x ($n \in \mathbb{N}$). In this paper, we show a central limit theorem and a law of the iterated logarithm for the random variables sequence $\{k_n, n \geq 1\}$, which generalize the results of Faivre [8] and Wu [31] respectively from $\beta = 10$ to any $\beta > 1$.

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1. Introduction

Let $\beta > 1$ be a real number and $T_\beta : [0, 1) \longrightarrow [0, 1)$ be the *β-transformation* defined as

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Then every $x \in [0, 1)$ can be uniquely expanded into a finite or infinite series, i.e.,

$$x = \frac{\varepsilon_1(x)}{\beta} + \frac{\varepsilon_2(x)}{\beta^2} + \cdots + \frac{\varepsilon_n(x)}{\beta^n} + \cdots, \tag{1.1}$$

where $\varepsilon_1(x) = \lfloor \beta x \rfloor$ and $\varepsilon_{n+1}(x) = \varepsilon_1(T_\beta^n x)$ for all $n \geq 1$. We call the representation (1.1) the *β-expansion* of x denoted by $(\varepsilon_1(x), \varepsilon_2(x), \cdots, \varepsilon_n(x), \cdots)$ and $\varepsilon_n(x), n \geq 1$ the *digits* of x. Such an expansion was first introduced by Rényi [27], who proved that there exists a unique T_β-invariant measure equivalent to the Lebesgue measure P when β is not an integer; while it is known that the Lebesgue measure is T_β-invariant when β is an integer. Furthermore, Gel'fond [12] and Parry [24] independently found the density formula for this invariant measure with respect to (w.r.t.) the Lebesgue measure. The arithmetic and metric properties of β-expansion were studied extensively in the literature, such as [2,6,9,11,14,18,19,28,29] and the references therein.

Now we turn our attention to continued fraction expansions. Let $T : [0, 1) \longrightarrow [0, 1)$ be the *Gauss transformation* given by

$$Tx = \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor, & \text{if } x \in (0, 1); \\ 0, & \text{if } x = 0. \end{cases}$$

Then any real number $x \in [0, 1)$ can be written as

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots + \cfrac{1}{a_n(x) + \ddots}}}, \tag{1.2}$$

where $a_1(x) = \lfloor 1/x \rfloor$ and $a_{n+1}(x) = a_1(T^n x)$ for all $n \geq 1$. The form (1.2) is said to be the *continued fraction expansion* of x and $a_n(x)$, $n \geq 1$ are called the *partial quotients* of x. Sometimes we write the form (1.2) as $[a_1(x), a_2(x), \cdots, a_n(x), \cdots]$. For any $n \geq 1$, we denote by $\frac{p_n(x)}{q_n(x)} := [a_1(x), a_2(x), \cdots, a_n(x)]$ the n-th *convergent* of x, where $p_n(x)$ and $q_n(x)$ are relatively prime. Clearly these convergents are rational numbers and $p_n(x)/q_n(x) \to x$ as $n \to \infty$ for all $x \in [0, 1)$. More precisely,