Journal of Number Theory 163 (2016) 570–595

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Unified approaches to the approximations of the gamma function

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A R T I C L E I N F O

Article history:
Received 4 October 2015
Received in revised form 23 December 2015
Accepted 23 December 2015
Available online 3 February 2016
Communicated by David Goss

MSC:
33B15
41A60

Keywords:
Approximations
Gamma function
Bell polynomials

A B S T R A C T

In this paper, by the exponential complete Bell polynomials, we establish two general approximations for the gamma function, where the coefficients in the series of the approximations can be determined by recurrences. These two general approximations include as special cases some well-known results such as those under the names De Moivre, Gosper, Gosper–Smith, Laplace, Luschny, Ramanujan and Wehmeier, and some recently published results such as those due to Batir, Chen, Lu, Mortici, Nemes, Paris, et al. By these two general approximations, we give unified approaches to dealing with many known approximations of the gamma function and to establishing new ones.

1. Introduction

The gamma function $\Gamma(x)$ is one of the most important functions in mathematics and has many important applications in various branches of science. Many researches are devoted to establishing approximation formulas for the gamma function and the related factorial function. For example, the well-known Stirling formula is

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http://dx.doi.org/10.1016/j.jnt.2015.12.016
0022-314X/

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x \exp\left(\sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right)$$
$$= \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x \exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \cdots\right), \quad (1.1)$$

as $x \to \infty$, where B_n are the Bernoulli numbers and $|\arg x| \le \pi - \delta < \pi$ $(0 < \delta \le \pi)$ (see, for example, [1, Chapter 6]), the Laplace formula is

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51\,840x^3} - \frac{571}{2\,488\,320x^4} + \cdots\right),$$

as $x \to \infty$, and the Ramanujan formula is

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} - \frac{11}{1920x^4} + \frac{79}{26\,880x^5} + \cdots\right)^{\frac{1}{6}},$$

as $x \to \infty$.

For an overview of approximations for the factorial function and gamma function, the readers are referred to Luschny's webpage [21], where he lists many approximations, including the Stirling formula, the Laplace formula, the Ramanujan formula and some other famous ones, such as the De Moivre formula, the Gosper formula, the Stieltjes formula and the Wehmeier formula. Moreover, the webpage also presents some results published recently, for example, the formulas due to Burić and Elezović [6] and Nemes [33,34].

There are so many approximations for the gamma function that have been proposed. Therefore, it is natural to find unified approaches to dealing with these approximations and to establishing new ones.

Mortici [22,23] introduced a family of approximations:

$$\Gamma(x+1) \sim \sqrt{2\pi\mathrm{e}} \cdot \mathrm{e}^{-a}\left(\frac{x+a}{\mathrm{e}}\right)^{x+\frac{1}{2}}, \quad x \to \infty, \quad (1.2)$$

where $a \in [0,1]$, which gives

$$\sqrt{\frac{2\pi}{x+1}}\left(\frac{x+1}{\mathrm{e}}\right)^{x+1}, \quad \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x \quad \text{and} \quad \sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{\mathrm{e}}\right)^{x+\frac{1}{2}}$$

by setting $a = 1$, $a = 0$ and $a = 1/2$, respectively. It can be found that the leading terms of most of the approximations presented before are of these three forms, so Luschny defined them by kern0, kern1 and kern2 in his pseudo-code [21]. Mortici [24,26] also used (1.2) to study general approximations, which can be rewritten as