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Right triangle and parallelogram pairs with a common area and a common perimeter ☆

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ABSTRACT

By the theory of elliptic curves, we show that there are infinitely many integer right triangle and integer parallelogram pairs with a common area and a common perimeter.

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1. Introduction

In 1995, R.K. Guy [2] introduced a problem of Bill Sands, that asked for examples of an integer right triangle and an integer rectangle with a common area and a common perimeter, but there are no non-degenerate right triangle and rectangle pair. In the same paper he showed that there are infinitely many such isosceles triangle and rectangle pairs. In 2006, A. Bremner and R.K. Guy [1] proved that there are infinitely many such Heron triangle and rectangle pairs.

Now we consider another generalized problem to find integer right triangle and integer parallelogram pairs with a common area and a common perimeter. Suppose that such a right triangle has sides

$$(x, y, z) = (m^2 - n^2, 2mn, m^2 + n^2),$$

where $m > n$. The corresponding parallelogram has sides p, q and the intersection angle of them θ, where $0 < \theta \le \pi/2$. Noting the homogeneity of these sides, let m, n, p and q be positive rational numbers, then we have the Diophantine system

$$\begin{cases} mn(m^2 - n^2) = pq \sin\theta, \\ m^2 + mn = p + q. \end{cases} \tag{1.1}$$

From Eq. (1.1), $\sin\theta$ is a rational number. Let us consider

$$\sin\theta = \frac{2t}{t^2 + 1},$$

where t is a positive rational number. According to the relationship

$$f(t) = f(1/t) = \frac{2t}{t^2 + 1},$$

we set $0 < t \le 1$.

For $t = 1$, $\theta = \pi/2$, this is the case studied by R.K. Guy [2]. By the theory of elliptic curves, we prove

Theorem 1.1. *If the elliptic curve*

$$Y^2 = X^3 + t^2 X^2 - 4t^2(t^2 + 1)(t^2 - t + 1)X + 4t^4(t^2 + 1)^2$$

has a rational point $P = (X, Y)$ satisfying the condition

$$0 < X < 2t(t^2 + 1), \quad 0 < Y < tX + 2t^4 + 2t^2 \tag{1.2}$$

for some t, then there exist infinitely many integer right triangle and integer parallelogram pairs with a common area and a common perimeter for this t.