Journal of Number Theory 164 (2016) 191–207

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

# Level 16 analogue of Ramanujan's theories of elliptic functions to alternative bases

Dongxi Ye

*Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706, USA*

A R T I C L E   I N F O

*Article history:*
Received 24 November 2015
Received in revised form 15 December 2015
Accepted 15 December 2015
Available online 8 February 2016
Communicated by David Goss

*MSC:*
primary 11F11, 11F20

*Keywords:*
Dedekind's eta function
Eisenstein series
Modular equation
Ramanujan's theories of elliptic functions to alternative bases
Theta functions

A B S T R A C T

The analogous theory for level 16 of Ramanujan's theories of elliptic functions to alternative bases is developed by studying the level 16 modular function

$$h(q) = q\prod_{j=1}^{\infty}\frac{(1-q^{16j})^2(1-q^{2j})}{(1-q^j)^2(1-q^{8j})}.$$



## 1. Introduction

In his famous paper [20], Ramanujan derived several remarkable series that converge to $1/\pi$, for example,

$$\frac{1}{\pi} = \frac{1}{16}\sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{42n+5}{2^{12n}}, \tag{1.1}$$

*E-mail address:* lawrencefrommath@gmail.com.

http://dx.doi.org/10.1016/j.jnt.2015.12.017
0022-314X/

by studying

$$\left(\frac{2K}{\pi}\right)^2 = {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1\end{matrix}; 4k^2k'^2\right), \tag{1.2}$$

where $k$, $k'$ and $K$ are, respectively, the modulus, complementary modulus, and complete elliptic integral of the first kind, from Jacobi's theory of elliptic functions. Identity (1.2) can be closely related to modular forms of level 4 via the following reformulation [6, Theorem 3.5]

$$q\frac{d}{dq}\log\left(\frac{w}{1-16w}\right) = \sum_{j=0}^{\infty}\binom{2j}{j}^3 (w(1-16w))^j, \tag{1.3}$$

where

$$w = q\prod_{j=1}^{\infty}\frac{(1-q^j)^8(1-q^{4j})^{16}}{(1-q^{2j})^{24}} \tag{1.4}$$

and, here and throughout the remainder of this work, $|q| < 1$. After remarking that "There are corresponding theories in which $q$ is replaced by one or other of the functions

$$q_1 = \exp\left(-\pi\sqrt{2}K_1'/K_1\right), \quad q_2 = \exp\left(-2\pi K_2'/(K_2\sqrt{3})\right)$$

and

$$q_3 = \exp\left(-2\pi K_3'/K_3\right),$$

where

$$K_1 = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; k^2\right),$$
$$K_2 = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; k^2\right),$$
$$K_3 = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; k^2\right)."$$

Ramanujan then offered 16 further formulas for $1/\pi$ that arise from modular forms of levels 1, 2 and 3, but he provides no details for his proofs. This motivated Berndt et al. [4] to analyze and develop the corresponding theories for levels 1, 2 and 3, which are now collectively known as "Ramanujan's theories of elliptic functions to alternative bases". We also refer the reader to [9] for a different analysis.

Under the inspiration of Ramanujan's work, analogous theories have been developed for several other levels. Details and further references are summarized in the following table.