Journal of Number Theory 164 (2016) 299–302

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A generalization of Beurling's criterion for the Riemann hypothesis

Jongho Yang

Department of Mathematics, Korea University, Seoul 02841, Republic of Korea

ARTICLE INFO

Article history:
Received 3 December 2015
Received in revised form 21 January 2016
Accepted 21 January 2016
Available online 4 March 2016
Communicated by David Goss

MSC:
11Mxx

Keywords:
Riemann hypothesis
Nyman–Beurling theorem
Bercovici–Foias theorem

ABSTRACT

It is known that the Riemann hypothesis holds if and only if the function $\chi_{(0,1)}$ can be approximated by linear combinations of u_α in $L^2(0,1)$. Here $u_\alpha(x)$ is defined by $[\alpha/x] - \alpha[1/x]$ for $0 < \alpha < 1$. In this note we generalize the Beurling's equivalent condition by replacing the function $\chi_{(0,1)}$ with $\chi_{(a,b)}$ for any $0 \leq a < b \leq 1$.

1. Introduction

Let $\mathcal{N}$ be the linear space generated by the functions u_α on $(0,1)$ defined by

$$u_\alpha(x) := [\alpha/x] - \alpha[1/x]$$

for $0 < \alpha < 1$. And the closure of $\mathcal{N}$ in $L^p(0,1)$ is denoted by $\mathcal{N}^p$.

Due to Nyman and Beurling, the density of $\mathcal{N}$ is closely related to the location of zeros of the Riemann zeta function. In [7], Nyman showed that the Riemann hypothesis

E-mail address: cachya@korea.ac.kr.

http://dx.doi.org/10.1016/j.jnt.2016.01.024

is true if and only if $\mathcal{N}^2 = L^2(0,1)$. Later, Beurling gave a generalization of Nyman's result in [5].

Theorem 1.1. *For $1 < p < \infty$, the following statements are equivalent.*

(i) The Riemann zeta function $\zeta(s)$ has no zeros in the half-plane $\Re s > 1/p$.
(ii) $\mathcal{N}^p = L^p(0,1)$.
(iii) $\chi_{(0,1)} \in \mathcal{N}^p$, where χ is the characteristic function.

There is exhaustive list of results in a density condition (ii). For example we refer [1,2,6]. In this note we generalize the condition (iii) as follows.

Theorem 1.2. *The Riemann hypothesis is true if and only if the space $\mathcal{N}^2$ contains a function $\chi_{(a,b)}$ for any $0 \le a < b \le 1$.*

2. Proof and questions

We define a compression operator T_s with $0 < s < 1$ by

$$T_s f(x) := \begin{cases} f(x/s), & 0 < x \le s \\ 0, & s < x < 1 \end{cases}$$

for $f \in \mathcal{N}^2$. It is known that the space $\mathcal{N}^2$ is closed under the operator T_s for any $0 < s < 1$. See [5] for example.

To prove the Theorem 1.2, we use the Bercovici–Foias theorem which characterizes the complement space of $\mathcal{N}$ in $L^2(0,1)$ as functions induced by zeros of the Riemann zeta function on the half-plane with $\Re s > 1/2$. More precisely, they showed the following theorem in [4].

Theorem 2.1. *Let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $L^2(0,1)$. Then we have*

$$\mathcal{N}^\perp = span_{L^2(0,1)}\{t \to t^{s-1} \log^k t, \zeta(s) = 0 \text{ with } \Re s > 1/2\},$$

where $0 \le k \le$ multiplicity of s.

See also [3,8] for related materials. Now we are ready to prove the Theorem 1.2.

Proof of Theorem 1.2. If the Riemann hypothesis holds, the space $\mathcal{N}^2 = L^2(0,1)$ by the Beurling's result. So $\mathcal{N}^2$ contains a function $\chi_{(a,b)}$ for any $0 \le a < b \le 1$.

To prove the other direction we first consider a case of $a = 0$. Assume that a function $\chi_{(0,b)}$ is in $\mathcal{N}^2$ for some $0 < b \le 1$. By applying the operator T_s to $\chi_{(0,b)}$, we have that $\chi_{(0,b')}$ is in $\mathcal{N}^2$ for any $0 < b' < b$. So any function x^{s-1} with $\Re s > 1/2$ cannot be in $\mathcal{N}^\perp$. By the Theorem 2.1, the zeta function has no non-trivial zeros.