Journal of Number Theory 164 (2016) 323–342

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Asymptotics for cuspidal representations by functoriality from $GL(2)$

Huixue Lao [a], Mark McKee [b], Yangbo Ye [b,*]

[a] *School of Mathematical Sciences, Shandong Normal University, Jinan, Shandong 250014, China*
[b] *Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, USA*

ARTICLE INFO

Article history:
Received 8 June 2015
Received in revised form 16 January 2016
Accepted 22 January 2016
Available online 3 March 2016
Communicated by David Goss

MSC:
11F70
11F66
11F30

Keywords:
Cuspidal representation
Functoriality
$GL(2)$
Fourier coefficient
Asymptotic expansion

ABSTRACT

Let π be a unitary automorphic cuspidal representation of $GL_2(\mathbb{Q}_{\mathbb{A}})$ with Fourier coefficients $\lambda_\pi(n)$. Asymptotic expansions of certain sums of $\lambda_\pi(n)$ are proved using known functorial liftings from GL_2, including symmetric powers, isobaric products, exterior squares, and base change. These asymptotic expansions are manifestation of the underlying functoriality and reflect value distribution of $\lambda_\pi(n)$ on integers, squares, cubes and fourth powers.

* Corresponding author.
E-mail addresses: lhxsdnu@163.com (H. Lao), mark-mckee@uiowa.edu (M. McKee), yangbo-ye@uiowa.edu (Y. Ye).

http://dx.doi.org/10.1016/j.jnt.2016.01.008
0022-314X/

1. Introduction

Let $\pi = \otimes \pi_p$ be a unitary automorphic irreducible cuspidal representation of $GL_2(\mathbb{Q}_\mathbb{A})$ with Fourier coefficients $\lambda_\pi(n)$. For each finite p for which π_p is unramified we can associate to π_p Satake parameters $\alpha_\pi(p), \beta_\pi(p) \in \mathbb{C}$ such that

$$\lambda_\pi(p) = \alpha_\pi(p) + \beta_\pi(p), \quad |\alpha_\pi(p)\beta_\pi(p)| = 1.$$

For ramified π_p, one can still choose $\alpha_\pi(p), \beta_\pi(p) \in \mathbb{C}$, by allowing some of them to be zero, such that the finite-part automorphic L-function attached to π is

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s} = \prod_{p<\infty} \left(1 - \alpha_\pi(p)p^{-s}\right)^{-1} \left(1 - \beta_\pi(p)p^{-s}\right)^{-1} \tag{1.1}$$

for $\mathrm{Re}\, s > 1$.

If π is associated with a holomorphic cusp form for a congruence subgroup of $SL_2(\mathbb{Z})$, the Ramanujan bound was proved by Deligne [4]

$$|\alpha_\pi(p)| = |\beta_\pi(p)| = 1$$

for p with π_p being unramified. For other cases the best known bound toward the Ramanujan conjecture is

$$|\alpha_\pi(p)|, |\beta_\pi(p)| \le p^{\frac{7}{64}}, \tag{1.2}$$

which is due to Kim and Sarnak [12].

Let us review known functoriality from an automorphic cuspidal representation π of $GL_2(\mathbb{Q}_\mathbb{A})$ (cf. Kim [11]).

1.1. The symmetric square lift. By Gelbart and Jacquet [5] $Sym^2\pi$ is an automorphic representation of $GL_3(\mathbb{Q}_\mathbb{A})$. It is cuspidal if and only if π is not monomial. Here π is monomial which means that there is a nontrivial grössencharacter ω so that $\pi \cong \pi \otimes \omega$. This in turn means that π corresponds to a dihedral Galois representation $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

1.2. The symmetric cube. Kim and Shahidi [14] proved that $Sym^3\pi$ is an automorphic representation of $GL_4(\mathbb{Q}_\mathbb{A})$. In addition, $Sym^3\pi$ is cuspidal if and only if π does not correspond to a dihedral or tetrahedral Galois representation. We recall π is tetrahedral means the π is not monomial and there exists a grössencharacter $\eta \neq 1$ so that $Ad^2\pi \cong Ad^2\pi \otimes \eta$. Here $Ad^2\pi = Sym^2\pi \otimes \omega_\pi^{-1}$, where ω_π is the central character of π.

1.3. The symmetric fourth power. By Kim [10] $Sym^4\pi$ is an automorphic representation of $GL_5(\mathbb{Q}_\mathbb{A})$. It is either cuspidal or unitarily induced from cuspidal representations of $GL_2(\mathbb{Q}_\mathbb{A})$ and $GL_3(\mathbb{Q}_\mathbb{A})$. More precisely, $Sym^4\pi$ is cuspidal unless π corresponds to either a dihedral, tetrahedral, or octahedral Galois representation (Kim and Shahidi [13]). Here π is octahedral means $Ad^3\pi$ is cuspidal, but there exists a nontrivial quadratic character μ so that $Ad^3\pi \cong Ad^3\pi \otimes \mu$.