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Congruences and asymptotics of Andrews' singular overpartitions

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ABSTRACT

Recently, *singular overpartitions* were defined and studied by G.E. Andrews. He showed that such partitions can be enumerated by $\overline{C}_{k,i}(n)$, the number of overpartitions of n such that no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. Andrews proved some congruences for $\overline{C}_{3,1}(n) \pmod 3$. The author, M.D. Hirschhorn and J.A. Sellers found infinite families of congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{4,1}(n)$, $\overline{C}_{6,1}(n)$ and $\overline{C}_{6,2}(n)$. Z. Ahmed and N.D. Baruah obtained some new congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{8,2}(n)$, $\overline{C}_{12,2}(n)$, $\overline{C}_{12,4}(n)$, $\overline{C}_{24,8}(n)$ and $\overline{C}_{48,16}(n)$. In this paper, we prove some new congruences for $\overline{C}_{3,1}(n)$ and $\overline{C}_{4,1}(n)$ modulo powers of 2 and congruences of $\overline{C}_{k,i}(n)$ for a family of pairs k, i. We also obtain an asymptotic formula for $\overline{C}_{k,i}(n)$ as n tends to infinity.

1. Introduction

S. Corteel and J. Loverjoy [9] introduced overpartitions ten years ago. An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. Recently, G.E. Andrews [3] introduced singular

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overpartitions. To describe such partitions, we recall the Frobenius symbol is a two-rowed array

$$\begin{pmatrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_r \end{pmatrix},$$

where $\sum(a_i + b_i + 1) = n$ and $a_1 > a_2 > \cdots > a_r \geq 0$, $b_1 > b_2 > \cdots > b_r \geq 0$. There is a natural mapping that reveals a one-to-one correspondence between the Frobenius symbols for n and the ordinary partitions of n. "Singular overpartitions" are Frobenius symbols for n with at most one overlined entry in each row. More precisely, for two positive integers k and i, a column $\begin{matrix} a_j \\ b_j \end{matrix}$ in a Frobenius symbol is (k,i)-*positive* if $a_j - b_j \geq k - i - 1$ and (k,i)-*negative* if $a_j - b_j \leq -i + 1$. If $-i + 1 < a_j - b_j < k - i + 1$, then we say the column is (k,i)-*neutral*. Two columns have the same parity if they are both (k,i)-*positive* or (k,i)-*negative*. We can divide the Frobenius symbol into (k,i)-*block* such that all the entries in each block have either the same (k,i)-*parity* or are (k,i)-*neutral*. The first non-neutral column in each parity block is called the *anchor* of the block. A (k,i)-parity block is neutral if all columns in it are neutral and a (k,i)-parity block is positive (resp. negative) if it contains no (k,i)-negative (resp. positive) columns.

A Frobenius symbol is (k,i)-singular if

(1) there are no overlined entries, or
(2) the one overlined entry on the top row occurs in the anchor of a (k,i)-positive block, or
(3) the one overlined entry on the bottom row occurs in an anchor of a (k,i)-negative block, and
(4) if there is one overlined entry in each row, then they occur in adjacent (k,i)-parity blocks.

Let $\overline{Q}_{k,i}(n)$ denote the number of such singular overpartitions for $1 \leq i < \frac{k}{2}$. G.E. Andrews proved that

$$\overline{Q}_{k,i}(n) = \overline{C}_{k,i}(n),$$

where $\overline{C}_{k,i}(n)$ is the number of overpartitions of n such that no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. Therefore, for $k \geq 3$ and $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{Q}_{k,i}(n) q^n &= \sum_{n=0}^{\infty} \overline{C}_{k,i}(n) q^n \\ &= \frac{(q^k; q^k)_\infty (-q^i; q^k)_\infty (-q^{k-i}; q^k)_\infty}{(q;q)_\infty} \end{aligned} \tag{1}$$