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Rational hyperbolic triangles and a quartic model of elliptic curves

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ABSTRACT

The family of Euclidean triangles having some fixed perimeter and area can be identified with a subset of points on a nonsingular cubic plane curve, i.e., an elliptic curve; furthermore, if the perimeter and the square of the area are rational, then the curve has rational coordinates and those triangles with rational side lengths correspond to rational points on the curve. We first recall this connection, and then we develop hyperbolic analogs. There are interesting relationships between the arithmetic on the elliptic curve (rank and torsion) and the family of triangles living on it. In the hyperbolic setting, the analogous plane curve is a quartic with two singularities at infinity, so the genus is still 1. We can add points geometrically by realizing the quartic as the intersection of two quadric surfaces. This allows us to construct nontrivial examples of rational hyperbolic triangles having the same inradius and perimeter as a given rational right hyperbolic triangle.

1. Introduction

Connections between families of triangles and elliptic curves have been long studied. Consider, for example, the famous congruent number problem: What rational numbers

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A are the area of a right triangle with rational side lengths? We call such A *congruent numbers*. Using similar triangles, we may assume A is a squarefree positive integer. For example, 6 is a congruent number since it is the area of the $(3, 4, 5)$-right triangle. Euler showed that 7 is a congruent number, but 1, 2, and 3, are not. It turns out that a positive rational number A is a congruent number if and only if the elliptic curve $y^2 = x^3 - A^2x$ has a rational point in the first quadrant; in fact, since the only torsion points on this elliptic curve have order dividing two, A being congruent here is equivalent to the group of rational points on the curve having positive rank. More generally, one can ask whether A is the area of a *rational triangle* (i.e., a triangle having rational side lengths) one of whose internal angles is some fixed value $\theta \in (0, \pi)$. Such numbers correspond to the existence of a certain kind of rational point on elliptic curves of the form $y^2 = x(x - A\lambda)(x + A/\lambda)$ where $\lambda = \sin(\theta)/(\cos(\theta) + 1)$ as found in Problem 3, Section 2, Chapter I of [Kob93].

Similarly, one can consider rational numbers A which are the area of a rational triangle having some fixed perimeter instead of a fixed internal angle. Again, the existence of such a triangle corresponds to the existence of a rational point on an elliptic curve, namely, a rational point in the first quadrant on the curve $s^2xy - A^2 = sxy(x + y)$ where s is the *semiperimeter* defined to be half the perimeter. We will recall this connection in Section 2. Note that the area A of a triangle is determined via Heron's formula $A = rs$ where r is the inradius, so we are led to study rational points on the curves $C_{r,s}\colon s(xy - r^2) = xy(x + y)$, which are similar to those studied in [CG] and/or [GM06]. From the equation for $C_{r,s}$, it is clear that similar triangles do, in fact, define isomorphic curves and that the family of curves can be parameterized by $k = s/r$. We will keep the parameters r, s separate, however, to motivate the hyperbolic analogs which have no chance of exploiting similarity since similar hyperbolic triangles are actually congruent.

In general, any triangle gives rise to a point on $C_{r,s}$ in the first quadrant where r is the triangle's inradius and s is its semiperimeter. (Here $s \geq 3\sqrt{3}r > 0$, and conversely, for any pair of real numbers r, s satisfying this inequality, there is a triangle with these parameters.) Such a curve $C_{r,s}$ will be an elliptic curve provided the triangle we started with was not equilateral or, equivalently, $s > 3\sqrt{3}r$. Moreover, if the given triangle is rational, then the point will have rational coordinates and $C_{r,s}$ will have rational coefficients since here $A^2 \in \mathbb{Q}$ even though A itself is not necessarily rational.

In particular, when $s, r^2 \in \mathbb{Q}$ and $s > 3\sqrt{3}r > 0$, one can ask questions about how the structure of the Mordell–Weil group $C_{r,s}(\mathbb{Q})$ is related to the family of rational triangles with parameters r, s. For example, "When do the points coming from rational triangles have infinite order?" It turns out that many Pythagorean triples give rise to rational points of infinite order (see Theorem 4 below), a fact which comes from the observation that points corresponding to triangles cannot have odd order. There can be points of even order coming from triangles, and such a point will have order 2 or 6 if and only if the corresponding triangle is isosceles.

We can also play the same game for hyperbolic triangles. The natural curve we get for hyperbolic triangles of a fixed inradius r and fixed semiperimeter s, however, is a quartic curve