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Fast computation of the partition function

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In this paper, the author provides a method to compute the values of Euler's partition function $p(n)$ that requires only the values of $p(k)$ with $k \leqslant n/2$. This method is combined with Ewell's recurrence relation for the partition function $p(n)$ to obtain a simple and fast computation of the value of $p(n)$.

1. Introduction

Recall [1] that a *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n, i.e.,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

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The partition function $p(n)$ counts the number of such partitions of n. For example, the partitions of 4 are:

$$4, \qquad 3+1, \qquad 2+2, \qquad 2+1+1 \qquad \text{and} \qquad 1+1+1+1.$$

Therefore, $p(4)=5$.

Euler [3] began the mathematical theory of partitions in 1748 by discovering the generating function of the partition function $p(n)$,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

is the q-shifted factorial, with $(a;q)_0=1$. Shortly after that, he gives an efficient method of computing the partition function $p(n)$, i.e.,

$$\sum_{k=-\infty}^{\infty} (-1)^k p\left(n-\frac{k(3k-1)}{2}\right) = \delta_{0,n}, \tag{1}$$

where $p(n)=0$ for any negative integer n and $\delta_{i,j}$ is the Kronecker delta. As we can see in [5, p. 286], this recurrence relation has about $\sqrt{8n/3}$ terms. In fact, computing the value of $p(n)$ with this relation requires all the values of $p(k)$, $k<n$.

Computing the values of the partition function $p(n)$ has been of interest to mathematicians. Over the years, mathematicians have given other recurrence relations for the partition function $p(n)$. From these, we remark a linear recurrence relation [4, Theorem 1.2] that has less than $\sqrt{2n}$ terms, i.e.,

$$p(n) + 2\sum_{k=1}^{\infty}(-1)^k p(n-2k^2) - \sum_{k=0}^{\infty} p\left(\frac{n}{4}-\frac{k(k+1)}{8}\right) = 0, \tag{2}$$

where $p(x)=0$ if x is not an integer. In other words, this recurrence is more efficient than Euler's recurrence. However, in this case computing the value of $p(n)$ requires all the values of $p(k)$, $k<n-1$.

There is a better way to compute an isolated value of $p(n)$, due to Hardy–Ramanujan–Rademacher formula

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} A_k(n) \frac{d}{dn}\left(\frac{1}{\sqrt{n-\frac{1}{24}}} \sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right]\right),$$