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A more accurate approximation for the gamma function

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ABSTRACT

In this paper, we establish a double inequality for the gamma function, from which we deduce the following approximation formula:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}},$$

$$x \to \infty,$$

which is more accurate than the Burnside, Gosper, Ramanujan, Windschitl, and Nemes formulas. We develop the previous approximation formula to produce an asymptotic expansion.

1. Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \in \mathbb{N} := \{1, 2, \dots\} \tag{1.1}$$

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has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n! \sim \text{constant} \cdot \sqrt{n}(n/e)^n$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

Stirling's series for the gamma function is given (see [1, p. 257, Eq. (6.1.40)]) by

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right) \tag{1.2}$$

as $x \to \infty$, where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi.$$

The following asymptotic formula is due to Laplace

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51\,840x^3} - \frac{571}{2\,488\,320x^4} + \cdots\right) \tag{1.3}$$

as $x \to \infty$ (see [1, p. 257, Eq. (6.1.37)]). The expression (1.3) is sometimes incorrectly called Stirling's series (see [19, pp. 2–3]). Stirling's formula is in fact the first approximation to the asymptotic formula (1.3).

Inspired by (1.1), Burnside [10] found a slightly more accurate approximation than Stirling's formula as follows:

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}. \tag{1.4}$$

A much better approximation is the following the Gosper formula [20]:

$$n! \sim \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^n. \tag{1.5}$$

The formulas (1.4) and (1.5) have motivated a large number of research papers; see [6, 7,9,17,26–29,31–39,41–43,51,52].