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The optimal constants of the mixed (ℓ_1, ℓ_2)-Littlewood inequality

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ABSTRACT

Text. In this note, among other results, we find the optimal constants of the generalized Bohnenblust–Hille inequality for m-linear forms over $\mathbb{R}$ and with multiple exponents $(1, 2, \ldots, 2)$, sometimes called mixed (ℓ_1, ℓ_2)-Littlewood inequality. We show that these optimal constants are precisely $(\sqrt{2})^{m-1}$ and this is somewhat surprising since a series of recent papers have shown that similar constants have a sublinear growth. This result answers a question raised by Albuquerque et al. in a paper published in 2014 in the *Journal of Functional Analysis.*

Video. For a video summary of this paper, please visit https://youtu.be/KnKtjbvsbW0.

1. Introduction

In the recent years a lot of papers (see, for instance, [4,6,9,15] and the references therein) have been dedicated to the search of best (or even optimal constants) for a class of famous inequalities, including the Littlewood's 4/3 inequality, the Bohnenblust–Hille inequality and the multilinear Hardy–Littlewood inequality (see [5,12,13]). The search

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of these constants, besides its intrinsic interest, have been shown to be important in different areas of Mathematics and even in Physics (see [6,14]). In this paper we find the optimal constants of a class of inequalities that encompasses the sometimes called mixed (ℓ_1, ℓ_2)-Littlewood inequality, which plays an important role in the recent development of the theory related to the Bohnenblust–Hille inequality.

The Khinchine inequality (see [8]) asserts that for all $0 < p < \infty$, there exist positive constants A_p and B_p such that

$$A_p \left(\sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{n=1}^{N} a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p \left(\sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}} \tag{1.1}$$

for every positive integer N and all real scalars $a_1, \ldots, a_N$ (here, r_n denotes the *n-th* Rademacher function, which is defined in $[0,1]$ by $r_n(t) = sgn\left(\sin 2^{n+1}\pi t\right)$).

The optimal constants of the Khinchine inequality are known. It is simple to observe that the optimal value of A_p is 1 for all $p \geq 2$ and $B_p = 1$ for all $p \leq 2$. For real scalars, U. Haagerup (see [11]) proved that the optimal constants A_p are (see also [8, page 23])

$$A_p = \frac{1}{\sqrt{2}} \left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}} \right)^{\frac{1}{p}}, \quad \text{for } 1.85 \approx p_0 < p < 2 \tag{1.2}$$

and

$$A_p = 2^{\frac{1}{2} - \frac{1}{p}}, \quad \text{for } 1 \leq p \leq p_0 \approx 1.85. \tag{1.3}$$

The exact definition of p_0 is the following: $p_0 \in (1,2)$ is the unique real number satisfying

$$\Gamma\left(\frac{p_0+1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Note that the Khinchine inequality tells us that

$$\left(\int_0^1 \left| \sum_{n=1}^{N} a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p A_r^{-1} \left(\int_0^1 \left| \sum_{n=1}^{N} a_n r_n(t) \right|^r dt \right)^{\frac{1}{r}} \tag{1.4}$$

regardless of the $0 < p, r < \infty$. From now on, as usual, c_0 denotes Banach space, endowed with the sup norm, of the sequences of scalars converging to zero. If $U : c_0 \times c_0 \to \mathbb{R}$ is a bilinear form, from the Khinchine inequality (and noting that from (1.3) we have $A_1 = 2^{-1/2}$) we have, for all positive integers N,

$$\sum_{i=1}^{N} \left(\sum_{j=1}^{N} |U(e_i, e_j)|^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \sum_{i=1}^{N} \int_0^1 \left| \sum_{j=1}^{N} r_j(t) U(e_i, e_j) \right| dt$$