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Combinatorial interpretations of a recent convolution for the number of divisors of a positive integer

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ABSTRACT

In this paper, we give a refined form of a recent factorization of Lambert series. This result allows us to prove new connections between partitions and divisors of positive integers, such as a new formula for the number of divisors of a positive integer as a convolution. Three recurrence relations for computing the number of partitions of a positive integer into distinct parts are rediscovered in this context.

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1. Introduction

The number of divisors of a given positive integer n is denoted in this paper by $\tau(n)$. The function $\tau(n)$ is the coefficient of q^n in the expansion of Lambert series in a power series,

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} \tau(n) q^n, \qquad |q| < 1.$$

Recently, Merca [4] provided the following factorization of Lambert series:

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{nq^{\binom{n+1}{2}}}{(q;q)_n}, \qquad |q| < 1,$$

where

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

is the q-shifted factorial, with $(a;q)_0 = 1$.

As a corollary of this result a finite discrete convolution for the function $\tau(n)$ was derived:

$$\tau(n) = \sum_{k=1}^{n} a(k) p(n-k), \tag{1}$$

where

$$a(n) = \sum_{k=1}^{\infty} (-1)^{k-1} k q(n,k),$$

$p(n)$ is the number of partitions of n, and $q(n,k)$ denotes the number of partitions of n into exactly k distinct parts ($q(0,k) = 1$ and $q(n,k) = 0$ for n negative). We remark that $a(n)$ is in fact a finite sum, since $q(n,k) = 0$ for $k > n$.

There are two natural questions related to this convolution:

1. Is it possible to have a much simpler expression for $a(n)$?
2. Is it possible to have a combinatorial interpretation for $a(n)$?

In this paper, we answer these questions (see for instance Corollary 5.1). New connections between partitions and divisors are derived in this context. As a consequence of these connections, we rediscover three linear recurrence relations for computing the number of partitions of a positive integer into distinct parts (see Corollaries 4.4, 4.5 and 4.6).