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Visual properties of generalized Kloosterman sums ☆

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ABSTRACT

For a positive integer m and a subgroup Λ of the unit group $(\mathbb{Z}/m\mathbb{Z})^\times$, the corresponding *generalized Kloosterman sum* is the function $K(a,b,m,\Lambda) = \sum_{u\in\Lambda} e(\frac{au+bu^{-1}}{m})$ for $a,b \in \mathbb{Z}/m\mathbb{Z}$. Unlike classical Kloosterman sums, which are real valued, generalized Kloosterman sums display a surprising array of visual features when their values are plotted in the complex plane. In a variety of instances, we identify the precise number-theoretic conditions that give rise to particular phenomena.

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1. Introduction

For a positive integer m and a subgroup Λ of the unit group $(\mathbb{Z}/m\mathbb{Z})^\times$, the corresponding *generalized Kloosterman sum* is the function

$$K(a,b,m,\Lambda) = \sum_{u \in \Lambda} e\left(\frac{au + bu^{-1}}{m}\right) \tag{1}$$

for $a, b \in \mathbb{Z}/m\mathbb{Z}$, in which $e(x) = \exp(2\pi i x)$ and u^{-1} denotes the multiplicative inverse of u modulo m. Classical Kloosterman sums arise when $\Lambda = (\mathbb{Z}/m\mathbb{Z})^\times$ [9].

Unlike their classical counterparts, which are real valued, generalized Kloosterman sums display a surprising array of visual features when their values are plotted in the complex plane; see Fig. 1. Our aim here is to initiate the investigation of these sums from a graphical perspective. In a variety of instances, we identify the precise number-theoretic conditions that give rise to particular phenomena.

Like classical Kloosterman sums, generalized Kloosterman sums enjoy a certain multiplicative property. If $m = m_1 m_2$, in which $(m_1, m_2) = 1$, $r_1 \equiv m_1^{-1} \pmod{m_2}$, $r_2 \equiv m_2^{-1} \pmod{m_1}$, $\omega_1 = \omega \pmod{m_1}$, and $\omega_2 = \omega \pmod{m_2}$, then

$$K\big(a,b,m,\langle\omega\rangle\big) = K\big(r_2 a, r_2 b, m_1, \langle\omega_1\rangle\big) K\big(r_1 a, r_1 b, m_2, \langle\omega_2\rangle\big). \tag{2}$$

This follows immediately from the Chinese Remainder Theorem. Consequently, we tend to focus on prime or prime power moduli; see Fig. 2. Since the group of units modulo an odd prime power is cyclic, most of our attention is restricted to the case where $\Gamma = \langle\omega\rangle$ is a cyclic group of units.

Additional motivation for our work stems from the fact that generalized Kloosterman sums are examples of supercharacters. The theory of supercharacters, introduced in 2008 by P. Diaconis and I.M. Isaacs [4], has emerged as a powerful tool in combinatorial representation theory. Certain exponential sums of interest in number theory, such as Ramanujan, Gauss, Heilbronn, and classical Kloosterman sums, arise as supercharacter values on abelian groups [1,5,7,6,8]. In the terminology of [1], the functions (1) arise by letting $n = m$, $d = 2$, and $\Gamma = \{\operatorname{diag}(u, u^{-1}) : u \in \Lambda\}$.

2. Hypocycloids

In what follows, we let ϕ denote the Euler totient function. If $q = p^\alpha$ is an odd prime power, then $(\mathbb{Z}/q\mathbb{Z})^\times$ is cyclic. Thus, for each divisor d of $\phi(q) = p^{\alpha-1}(p-1)$, there is a unique subgroup Λ of $(\mathbb{Z}/q\mathbb{Z})^\times$ of order d. In this case, we write

$$K(a,b,q,d) = \sum_{u^d = 1} e\left(\frac{au + bu^{-1}}{q}\right)$$

instead of $K(a,b,q,\Lambda)$.