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On the fourth power mean of the analogous general Kloosterman sum ☆

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ABSTRACT

Text. With the aids of elementary methods, the fourth power mean value of the analogous general Kloosterman sums $C(m, n, k, \chi; q)$ is studied, and an explicit formula is obtained. It shows that $C(m, n, k, \chi; q)$ enjoys good mean value distribution properties.

Video. For a video summary of this paper, please visit https://youtu.be/FZmy08BTpH8.

1. Introduction

Let $q \geq 3$ be a positive integer. For any integers m, n and k, the general k-th Kloosterman sum $S(m, n, k, \chi; q)$ is defined as follows:

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$$S(m,n,k,\chi;q)=\sum_{a=1}^{q}{}' \chi(a)e\left(\frac{ma^k+n\overline{a}^k}{q}\right),$$

where $\sum_{a=1}^{q}{}'$ denotes the summation over all a with $(a,q)=1$, $a\overline{a}\equiv 1(\bmod\ q)$, $e(y)=e^{2\pi iy}$.

This summation is very important, because it becomes the classical general Kloosterman sum

$$S(m,n,\chi;q)=\sum_{a=1}^{q}{}' \chi(a)e\left(\frac{ma+n\overline{a}}{q}\right)$$

if $k\equiv 1(\bmod\ \phi(q))$. And it is also a generalization of the classical Kloosterman sum

$$S(m,n;q)=\sum_{a=1}^{q}{}' e\left(\frac{ma+n\overline{a}}{q}\right).$$

Many authors have studied various properties of the above sums. For example, we know the estimate of $S(m,n;q)$ (see [4] or [2])

$$|S(m,n;q)|\le q^{\frac{1}{2}}d(q)(m,n,q)^{\frac{1}{2}},$$

where $d(n)$ is the divisor function, (m,n,q) is the greatest common divisor of m,n and q.

For arbitrary integer $q\ge 3$, we don't know how large $|S(m,n,k,\chi;q)|$ is. However, $S(m,n,k,\chi;q)$ enjoys good mean value distribution properties. For fixed integer n with $(n,q)=1$, Zhang [9] showed the identity (corrected by [8])

$$\sum_{\chi \bmod q}\sum_{m=1}^{q}|S(m,n,\chi;q)|^4=\phi^2(q)q^2d(q)\prod_{p^\alpha\|q}\left(1-\frac{1}{(\alpha+1)(p-1)}\right),$$

where $\phi(q)$ is the Euler function, and $\prod_{p^\alpha\|q}$ is the product over all prime divisors p of q with $p^\alpha|q$ and $p^{\alpha+1}\nmid q$.

For the general k-th Kloosterman sum $S(m,n,k,\chi;q)$, Liu and Zhang [7] proved the identity (the final result was corrected by us through amending a little computing mistake in Lemma 2.1 of [7])

$$\sum_{\chi \bmod q}\sum_{m=1}^{q}|S(m,n,k,\chi;q)|^4=\phi^2(q)q^2\prod_{p^\alpha\|q}(k,p-1)\left(2+\frac{(k,p-1)(\alpha p-p-\alpha)-2}{p}\right)$$

under the condition that $(nk,q)=1$. If p is an odd prime, α and k are positive integers, Guo, Geng, and Pan [5] considered the case $(k,q)>1$ and got the identity