Journal of Number Theory 160 (2016) 385–408

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Riesz-type criteria and theta transformation analogues

Atul Dixit[a,b], Arindam Roy[c,*], Alexandru Zaharescu[c,d]

[a] *Department of Mathematics, Tulane University, New Orleans, LA 70118, USA*
[b] *Department of Mathematics, Indian Institute of Technology Gandhinagar, Ahmedabad, GJ 382424, India*[1]
[c] *Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA*
[d] *Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania*

ARTICLE INFO

Article history:
Received 6 December 2014
Received in revised form 30 August 2015
Accepted 31 August 2015
Available online 22 October 2015
Communicated by David Goss

MSC:
primary 11M06
secondary 11M26, 33C15

Keywords:
Möbius function
Riemann zeta function
Dirichlet L-function
Riemann Hypothesis
Hypergeometric function
Ramanujan
Hardy
Ferrar

ABSTRACT

We give character analogues of a generalization of a result due to Ramanujan, Hardy and Littlewood, and provide Riesz-type criteria for Riemann Hypotheses for the Riemann zeta function and Dirichlet L-functions. We also provide analogues of the general theta transformation formula and of recent generalizations of the transformation formulas of W.L. Ferrar and G.H. Hardy for real primitive Dirichlet characters.

* Corresponding author.
E-mail addresses: adixit@tulane.edu, adixit@iitgn.ac.in (current) (A. Dixit), roy22@illinois.edu (A. Roy), zaharesc@math.uiuc.edu (A. Zaharescu).

[1] Current address.

http://dx.doi.org/10.1016/j.jnt.2015.08.005
0022-314X/

1. Introduction

In 1916, Riesz [16] gave the following equivalent criterion for the Riemann Hypothesis:

Let the function $F(x)$ be defined by

$$F(x) := \sum_{n=1}^{\infty} \mu(n) \frac{x}{n^2} e^{-x/n^2}.$$

The estimate $F(x) = O_\delta\left(x^{\frac{1}{4}+\delta}\right)$ for all $\delta > 0$ is a necessary and sufficient condition for the validity of the Riemann Hypothesis.

The following variant is due to Hardy and Littlewood [11, p. 156, Section 2.5]:

Consider the function

$$P(y) := \sum_{k=1}^{\infty} \frac{\mu(k)}{k} e^{-y/k^2} = \sum_{m=1}^{\infty} \frac{(-y)^m}{m!\zeta(2m+1)}. \tag{1.1}$$

Then, the estimate $P(y) = O_\delta\left(y^{-\frac{1}{4}+\delta}\right)$ as $y \to \infty$ for all positive values of δ is equivalent to the Riemann Hypothesis.

Their intuition and motivation came from a beautiful identity in Ramanujan's notebooks [15] (see also [3, p. 468, Entry 37]). The corrected version of this identity was given by Hardy and Littlewood [11, p. 156, Equation 2.516]. Various aspects of this identity have been presented by Berndt [3, p. 470], Bhaskaran [5], Paris and Kaminski [14, p. 143] and Titchmarsh [17, p. 219, Section 9.8]. A one-variable generalization of this identity was obtained in [8]. Motivated by these works and the aforementioned variant of Riesz's criterion, we establish the following theorem. An analogue for Dirichlet L-functions is given in Section 4.

Theorem 1.1. *Fix $z \in \mathbb{C}$. Consider the function*

$$\mathcal{P}_z(y) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-y/n^2} \cosh\left(\frac{\sqrt{y}z}{n}\right). \tag{1.2}$$

Then we have the following:

(1) *The Riemann Hypothesis implies $\mathcal{P}_z(y) = O_{z,\delta}\left(y^{-\frac{1}{4}+\delta}\right)$ as $y \to \infty$ for all $\delta > 0$.*

(2) (a) *If $z = 0$, the estimate $\mathcal{P}_z(y) = O_{z,\delta}\left(y^{-\frac{1}{4}+\delta}\right)$ as $y \to \infty$ for all $\delta > 0$ implies the Riemann Hypothesis.*

(b) *If $z \neq 0$ and $\arg(z) \neq \pm\frac{\pi}{4}$, the estimate $\mathcal{P}_z(y) = O_{z,\delta}\left(y^{-\frac{1}{4}+\delta}\right)$ as $y \to \infty$ for all $\delta > 0$ implies that $\zeta(s)$ has at most finitely many non-trivial zeros off the critical line.*