Journal of Number Theory 160 (2016) 409–417

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Salem numbers of trace −2, and a conjecture of Estes and Guralnick

James McKee *, Pavlo Yatsyna

Department of Mathematics, Royal Holloway, University of London, Egham Hill, Egham, Surrey, TW20 0EX, United Kingdom

A R T I C L E   I N F O

Article history:
Received 3 July 2015
Received in revised form 22 September 2015
Accepted 22 September 2015
Available online 2 November 2015
Communicated by David Goss

MSC:
11R06
11C99

Keywords:
Salem numbers
Minimal polynomials

A B S T R A C T

In 1993 Estes and Guralnick conjectured that any totally real separable monic polynomial with rational integer coefficients will occur as the minimal polynomial of some symmetric matrix with rational integer entries. They proved this to be true for all such polynomials that have degree at most 4.
In this paper, we show that for *every* $d \geq 6$ there is a polynomial of degree d that is a counterexample to this conjecture. The only case still in doubt is degree 5.
One of the ingredients in the proof is to show that there are Salem numbers of degree $2d$ and trace -2 for every $d \geq 12$.

1. Introduction

1.1. Salem numbers of trace −2

A *Salem number* is a real algebraic integer $\tau > 1$, conjugate to its reciprocal $1/\tau$, of degree at least 4, and with all conjugates other than τ and $1/\tau$ lying on the unit circle in the complex plane. See [20] for a recent survey. Smyth [19] considered the problem

* Corresponding author.
E-mail address: james.mckee@rhul.ac.uk (J. McKee).

http://dx.doi.org/10.1016/j.jnt.2015.09.019

of finding Salem numbers of negative trace, and found examples that had trace -1 of every (even) degree greater than or equal to 8. He asked how small the trace could be. McMullen [18] raised the question of whether or not there are *any* Salem numbers of trace less than -1, still none being known at that time. The first examples having trace -2 were found by McKee and Smyth [12], and indeed they showed that there are Salem numbers of every trace [13].

In this paper we shall show that there are Salem numbers of trace -2 for every (even) degree greater than or equal to 24: Proposition 1 below. The key new idea is to use an interlacing construction from [12] to produce a finite number of infinite families of Salem numbers that between them cover all sufficiently large degrees.

1.2. A conjecture of Estes and Guralnick

Let A be an integer symmetric matrix, and let $m_A(x)$ be its minimal polynomial. Then certainly $m_A(x)$ is a monic integer polynomial with all roots real. Moreover it is separable, since A is diagonalisable over $\mathbb{Q}$. In [7, page 84] Estes and Guralnick make the conjecture 'that any totally real separable monic integral polynomial can occur as the minimal polynomial of a symmetric integral matrix'. In support of this conjecture, they prove it to be true if the polynomial in question has degree at most 4.

Dobrowolski [4] showed that there are infinitely many counterexamples to the conjecture, by obtaining a lower bound on the discriminant of any polynomial that appears as the minimal polynomial of an integer symmetric matrix and noting that infinitely many totally real separable monic integral polynomials have a discriminant that is lower than his bound. The smallest known degree for any of his counterexamples is 2880.

McKee [10] found counterexamples that had much lower degrees, including three of degree 6. This was based on a classification of all integer symmetric matrices such that the difference between the largest and smallest eigenvalues is less than 4. Recently [17] we found a sharp lower bound for the trace of the minimal polynomial of an integer symmetric matrix, and used this to provide some further counterexamples to the Estes–Guralnick conjecture.

The current paper finds counterexamples for every degree greater than or equal to 6. All sufficiently large degrees are covered by minimal polynomials of numbers of the form $\tau+1/\tau+2$, where τ is a Salem number of trace -2. Smaller degrees are dealt with by *ad hoc* arguments. It is still not known whether the conjecture is true or false for degree-5 polynomials.

1.3. Statement of results

We now list the main results of the paper, and deduce some immediate corollaries, leaving the proofs of the main results until later. We start with an existence theorem for Salem numbers of trace -2 for all large enough degrees.