Journal of Number Theory 160 (2016) 463–477

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Class numbers of pure quintic fields

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ARTICLE INFO

Article history:
Received 1 January 2015
Received in revised form 5 August 2015
Accepted 9 September 2015
Available online 2 November 2015
Communicated by D. Burns

MSC:
primary 11R29
secondary 11R20, 11R21, 11R37

Keywords:
Pure quintic fields
Class numbers
Unit groups

ABSTRACT

Let m be a fifth power free integer greater than one. Let K be an algebraic number field generated by a fifth root of m over the rational number field. If m has a prime factor p congruent to -1 modulo five, the class number of K is a multiple of five.

1. Introduction

Let l be a prime, K a pure field of degree l, i.e., $K = \mathbb{Q}(m^{1/l})$ where m is an l-th power free integer greater than one, and L the Galois closure of K over the rational number field $\mathbb{Q}$. There is a question when the class number h_K of K is divisible by l. Genus theory gives an answer in the case $l = 2$. Honda [1] solved the cubic case and his method became a model of researches on this subject. Subsequently to Honda's study, Parry [5] studied the case $l = 5$ and found the difficulty in this case. He presented the relation formula between the class numbers of K and L:

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http://dx.doi.org/10.1016/j.jnt.2015.09.017
0022-314X/

$$5^5 h_L = c_m h_K^4,$$

where h_L is the class number of L and c_m is a divisor of 5^6. He also gave necessary and sufficient conditions for L to have the class number divisible by 5, and left the six cases unclear whether h_K is divisible by 5 or not (see Theorem IV of [5]). For instance, the divisibility remained unclear when m is a prime number p such that $p \equiv -1 \pmod 5$. Iimura [2] showed that there are infinitely many fields K with $5 \nmid h_K$ and $5 \mid h_L$ in this quintic case. For a general odd prime l, Parry and Walter [6] gave necessary and sufficient conditions for L to have the class number divisible by l. They also derived necessary conditions for K to have the class number divisible by l when the class number of the maximal real subfield of the l-th cyclotomic field is not divisible by l.

On the other hand, Ishida [3] showed that if l is an odd prime and m has a prime factor p with $p \equiv 1 \pmod l$, the class number of K is divisible by l. He showed that the composite field of K and the subfield of degree l of the p-th cyclotomic field is unramified over K, and then the divisibility follows from class field theory. Here, we pose the following conjecture:

Conjecture 1. *Let l be an odd prime greater than three and let p be a prime such that $p \equiv -1 \pmod l$. If an l-th power free positive integer m is divisible by p and $K = \mathbb{Q}(m^{1/l})$, then the class number of K is divisible by l.*

It is easy to see that Ishida's method does not work in this case. In this paper we prove this conjecture for $l = 5$, i.e.,

Theorem 1. *Let m be a fifth power free positive integer and let $K = \mathbb{Q}(m^{1/5})$. If m has a prime factor p with $p \equiv -1 \pmod 5$, then the class number of K is divisible by five.*

As a consequence, we make clear three of the six cases left by Parry. Our method is essentially based on an investigation of the Galois module structure of the unit group of L, but our description is solely devoted to the unit group of the maximal real subfield of L since it is sufficient for our purpose.

We will describe an outline of the proof briefly. Let $K = \mathbb{Q}(m^{1/5})$ where m is a fifth power free integer greater than one, and L the Galois closure of K over the rational number field $\mathbb{Q}$. Let L^+ be the maximal real subfield of L. In general, we show that $5 \mid h_K$ if and only if $5 \mid h_{L^+}$, where h_K and h_{L^+} are the class numbers of K and L^+ respectively. Further assume that $5 \nmid h_{L^+}$. Under this assumption, we determine a set of fundamental units of L^+ and investigate endomorphisms of the unit group of L^+. When m has a prime divisor p congruent to -1 modulo five, applying the investigation to an abstract unit constructed by totally ramified primes in the extension $L^+/\mathbb{Q}(\sqrt{5})$, we encounter a contradiction with our assumption that $5 \nmid h_{L^+}$.

Finally, we touch on the following useful theorem, which is used twice as $n = 1$ in this paper.