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Note on character sums of Hilbert cubes

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ABSTRACT

We give bounds for additive and multiplicative character sums of multiplicative and additive Hilbert cubes in prime fields.

1. Introduction

In 1892 Hilbert defined an affine d-dimensional cube which is nowadays called the Hilbert cube as follows: let $x_0, a_1 < a_2 < \cdots < a_d$ be any sequence of integers. The Hilbert cube is the set

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$$H(x_0, a_1, a_2, \dots, a_d) = \Big\{x_0 + \sum_{1 \le i \le d} \varepsilon_i a_i\Big\} \quad \varepsilon_i \in \{0, 1\}. \tag{1}$$

Hilbert cubes play an important role in the proof of Szemerédi's celebrated theorem, and many authors investigated them in different context (see e.g., [11,8,7,5]).

We can define a Hilbert cube of order $r \ge 1$; $r \in \mathbb{N}$ extending (1) by

$$H_r(x_0, a_1, a_2, \dots, a_d) = \Big\{x_0 + \sum_{1 \le i \le d} \varepsilon_i a_i\Big\} \quad \varepsilon_i \in \{0, 1, \dots, r\}. \tag{2}$$

When $r = 1$, we write shortly $H(x_0, a_1, a_2, \dots, a_d) = H_1(x_0, a_1, a_2, \dots, a_d)$.

We say that $\dim(H) := d$ is the dimension of H and $|H(x_0, a_1, a_2, \dots, a_d)|$ is its size. Let Δ, $0 < \Delta \le 1$ be a real parameter. We say that a cube $H =: H_r(x_0, a_1, a_2, \dots, a_d)$ is Δ-degenerate, if $\frac{\log_{r+1}|H|}{d} = \Delta$. (Here, and in what follows, $\log_{r+1} x$ means $\log x / \log(r+1)$.) When $\Delta = 1$, then $|H| = (r+1)^d$. In this case all terms in (2) are pairwise distinct and H is said to be non-degenerate.

We note that considering Hilbert cubes $H =: H_r(x_0, a_1, a_2, \dots, a_d)$ with $\Delta = \Delta(r)$ and $H =: H_{r+1}(x_0, a_1, a_2, \dots, a_d)$ with $\Delta = \Delta(r+1)$ the function $\Delta(r)$ could have a "jump". A simple example is when $H = H_1(0, 1, 2, \dots, 2^{d-1})$ is non-degenerate, while $H_2(0, 1, 2, \dots, 2^{d-1}) = H + H$ and $|H_2| \le 2|H_1|$.

For an arbitrary set $A \subseteq \mathbb{F}_p$ its *additive* energy is defined by

$$E_+(A) := \{(a_1, a_2, a_3, a_4) \in A^4 : a_1 + a_2 = a_3 + a_4\}$$

and its *multiplicative* energy is defined by

$$E_\times(A) := \{(a_1, a_2, a_3, a_4) \in A^4 : a_1 \cdot a_2 = a_3 \cdot a_4\}.$$

Let f be an arbitrary function from $\mathbb{F}_p^*$ to $\mathbb{C}$. Denote the Fourier transform (with respect to a multiplicative character) by

$$\widehat{f(u)} := \sum_{x \in \mathbb{F}_p^*} f(x) \chi_u(x)$$

where $\chi_u(x)$ is the multiplicative (Dirichlet) character; $\chi_u(x) = e^{\frac{2\pi i indx \cdot u}{p-1}}$ where *indx* is index of x (or it is sometimes said to be discrete logarithm). When $\chi \ne \chi_0$ is not the principal character, then let $\chi(0) = 0$. Recall that

$$\sum_{u \in \mathbb{F}_p^*} |\widehat{f(u)}|^2 = (p-1) \sum_{x \in \mathbb{F}_p^*} |f(x)|^2 \tag{3}$$

Let $g : \mathbb{F}_p \to \mathbb{C}$ and $x \in \mathbb{F}_p$. Denote the Fourier transform (with respect to an additive character) by