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Torsion of rational elliptic curves over quartic Galois number fields

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A R T I C L E I N F O

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A B S T R A C T

Let E be an elliptic curve defined over $\mathbb{Q}$, and let K be a number field of degree four that is Galois over $\mathbb{Q}$. The goal of this article is to classify the different isomorphism types of $E(K)_{\text{tors}}$.

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1. Introduction

Let E be an elliptic curve defined over $\mathbb{Q}$. Given a number field K, we may consider E as an elliptic curve defined over K and examine the structure of the points of E with coordinates in K, denoted $E(K)$. We have the following fundamental theorem describing the structure of $E(K)$:

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Theorem 1.1 *(Mordell–Weil). Let E be an elliptic curve over a number field K. The group of K-rational points, $E(K)$, is a finitely generated abelian group.*

By the fundamental theorem of finitely generated abelian groups it follows that, for any elliptic curve E over K, there exists an integer $r_K > 0$ depending on K such that

$$E(K) \cong E(K)_{\text{tors}} \oplus \mathbb{Z}^{r_K}$$

where $E(K)_{\text{tors}}$ is a finite group. We call r_K the rank of E over K, and we call $E(K)_{\text{tors}}$ the torsion subgroup of the E over K. A natural question is which groups can arise as torsion subgroups of elliptic curves over certain number fields.

In this paper we obtain a classification of the torsion subgroup of elliptic curves with rational coefficients over number fields K that are quartic Galois extensions of $\mathbb{Q}$. We separate the classification based on the isomorphism type of $\mathrm{Gal}(K/\mathbb{Q})$. If $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ we call K a cyclic quartic extension of $\mathbb{Q}$, and if $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ we call K a biquadratic extension of $\mathbb{Q}$.

The main results of this article are as follows:

Theorem 1.2. *Let $E/\mathbb{Q}$ be an elliptic curve, and let K be a quartic Galois extension of $\mathbb{Q}$. Then $E(K)_{tors}$ is isomorphic to one of the following groups:*

$$\begin{aligned}
&\mathbb{Z}/N_1\mathbb{Z}, && N_1 = 1, \ldots, 16, N_1 \neq 11, 14,\\
&\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N_2\mathbb{Z}, && N_2 = 1, \ldots, 6, 8,\\
&\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3N_3\mathbb{Z}, && N_3 = 1, 2,\\
&\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4N_4\mathbb{Z}, && N_4 = 1, 2,\\
&\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}, &&\\
&\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}. &&
\end{aligned}$$

Each of these groups, except for $\mathbb{Z}/15\mathbb{Z}$, appears as the torsion structure over some quartic Galois field for infinitely many (non-isomorphic) elliptic curves defined over $\mathbb{Q}$.

The proof of this theorem is broken up based on the structure of $\mathrm{Gal}(K/\mathbb{Q})$ and so, in fact, we have the following more specialized theorems:

Theorem 1.3. *Let $E/\mathbb{Q}$ be an elliptic curve, and let K be a quartic Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$. Then $E(K)_{tors}$ is isomorphic to one of the following groups:*

$$\begin{aligned}
&\mathbb{Z}/N_1\mathbb{Z}, && N_1 = 1, \ldots, 10, 12, 13, 15, 16,\\
&\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N_2\mathbb{Z}, && N_2 = 1, \ldots, 6, 8,\\
&\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}. &&
\end{aligned}$$

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